Back Reaction in Einstein Universe at Finite Temperature

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1. Motivation

1. Early studies

   The quantum field theory in curved space-time at finite temperature has been studied in these works (for example):

   Einstein’s field equation for a massless free scalar field was solved in curved space-time at finite temperature and the back-reaction was studied.

   “M.B. Altaie, M.R. Setare, Phys.Rev.D67:044018,2003, gr-qc/0301009“

   “T. Inagaki, K. Ishikawa and T. Muta,Prog.Theor.Phys.96:847,1996“

   “M.B. Altaie and J.S.Dowker, Phys.Rev.D18:1978“

2. Our study

   We study the theory of massive scalar field in D -dimensional curved space at finite temperature and evaluate the back reaction of this theory.
   Finite mass and coupling constant modify the back reaction and the evolution of the universe.

2. Model

- Space-time model

metric of the static space-time:

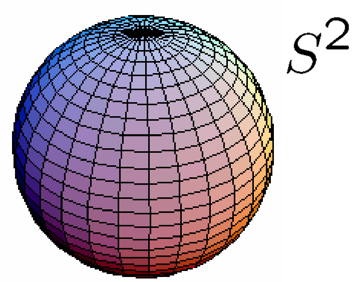
$$ds^2 = dt^2 \underset{\cdots\cdots}{+} a^2 d\sigma^2$$ (a :time independent scale factor)

…Euclidean analog of the static space-time called "Einstein universe"

- $T \otimes S^{D-1}$

$$d\sigma^2 = d\theta^2 + sin^2\theta d\Omega_{D-2}$$

$$R = (D-1)(D-2)\frac{1}{a^2} > 0$$

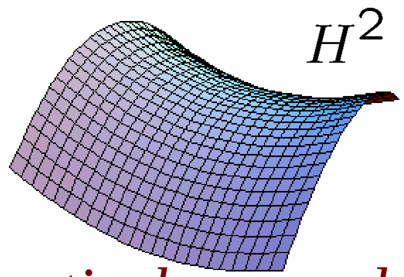


…positively curved space

- $T \otimes H^{D-1}$

$$d\sigma^2 = d\theta^2 + sinh^2\theta d\Omega_{D-2}$$

$$R = -(D-1)(D-2)\frac{1}{a^2} < 0$$



…negatively curved space

$d\Omega_{D-2}$:metric on a unit sphere S^{D-2}

- Lagrangian density of scalar field

$$\mathcal{L} = \frac{1}{2}\partial_{;}{}^{\mu}\phi\partial_{;\mu}\phi + \frac{1}{2}\phi^2(\mu_0^2 - \xi_0 R) - \frac{\lambda_0}{4!}\phi^4$$

$i\mu_0$:bare mass of the scalar field

λ_0 :bare coupling constant for the scalar self-interaction

R :scalar curvature of the space-time

ξ_0 :bare coupling constant between the scalar field and the scalar curvature

3. Analysis of the Model

- Effective potential (1-loop correction)

Effective potential in D-dimensional flat space:

$$V_0(\phi) = -\frac{\mu_0^2}{2}\phi^2 + \frac{\lambda_0}{4!}\phi^4 - \frac{\hbar}{2(4\pi)^{D/2}}\Gamma\left(-\frac{D}{2}\right)\left[\left(-\mu_0^2 + \frac{\lambda_0}{2}\phi^2\right)^{D/2} - (-\mu_0^2)^{D/2}\right]$$

Bare constants are renormalized by the renormalization conditions:

$$\left.\frac{\partial V_0}{\partial \phi^2}\right|_{\phi=0} \equiv -\mu_r^2 + \xi_0 R, \quad \left.\frac{\partial^4 V_0}{\partial \phi^2}\right|_{\phi=M} \equiv \lambda_r$$

◆ E

Effective potential in D-dimensional Einstein space:

$$V(\phi) = \frac{\xi_0}{2}R\phi^2 - \frac{\mu_0^2}{2}\phi^2 + \frac{\lambda_0}{4!}\phi^4 + \frac{\hbar\lambda_0}{4}\int_0^{\phi^2} dm^2 \underline{G(x,x;m)}$$

Two point function in Einstein universe

$\boxed{T \otimes S^{D-1}}$

$$G(x,y;m) = \frac{a^{3-D}}{(4\pi)^{(D-1)/2}}\int \frac{d\omega}{2\pi}e^{-i\omega(y-x)_4}\frac{\Gamma\left(\frac{D-2}{2}+i\alpha_S\right)\Gamma\left(\frac{D-2}{2}-i\alpha_S\right)}{\Gamma\left(\frac{D-1}{2}\right)}$$

$$\times F\left(\frac{D-2}{2}+i\alpha_S, \frac{D-2}{2}-i\alpha_S, \frac{D-1}{2}; \cos^2\left(\frac{\sigma}{2a}\right)\right)$$

$\boxed{T \otimes H^{D-1}}$

$$G(x,y;m) = \frac{a^{3-D}}{(4\pi)^{(D-1)/2}}\int \frac{d\omega}{2\pi}e^{-i\omega(y-x)_4}\cosh^{2-D-\alpha_H}\left(\frac{\sigma}{2a}\right)\frac{\Gamma\left(\frac{D-2}{2}+\alpha_H\right)\Gamma\left(\frac{1}{2}+\alpha_H\right)}{\Gamma(2\alpha_H+1)}$$

$$\times F\left(\frac{D-2}{2}+\alpha_H, \frac{1}{2}+\alpha_H, 2\alpha_H+1; \cosh^{-2}\left(\frac{\sigma}{2a}\right)\right)$$

$$\alpha_S \equiv \sqrt{f(\omega)a^2 + (D-1)(D-2)\xi_0 - \frac{(D-2)^2}{4}} \quad , \quad \alpha_H \equiv \sqrt{f(\omega)a^2 - (D-1)(D-2)\xi_0 + \frac{(D-2)^2}{4}}$$

$$f(\omega) \equiv \omega^2 - \mu_0^2 + \frac{\lambda_0}{2}m^2$$

"T. Inagaki, K. Ishikawa and T. Muta, Prog.Theor.Phys.96:847,1996"

◆ E

Effective potential in D-dimensional Einstein space at Finite Temperature:

Following the standard procedure of the imaginary time formalism,

$$\begin{cases} \displaystyle\int \frac{d\omega}{2\pi} \rightarrow \frac{1}{\beta}\sum_{n=-\infty}^{\infty} \\ \omega \rightarrow \omega_n \equiv \dfrac{2n}{\beta}\pi . \end{cases}$$

→

scalar two point function
at finite temperature is obtained

$\boxed{T \otimes S^{D-1}}$

$$V(\phi) = \frac{\xi_0}{2}R\phi^2 - \frac{\mu_0^2}{2}\phi^2 + \frac{\lambda_0}{4!}\phi^4 + \frac{\hbar\lambda_0}{4\beta}\frac{a^{3-D}}{(4\pi)^{(D-1)/2}}\Gamma\left(\frac{3-D}{2}\right) \times \int_0^{\phi^2} dm^2 \sum_{n=-\infty}^{\infty} \frac{\Gamma\left(\frac{D-2}{2}+i\alpha_S\right)\Gamma\left(\frac{D-2}{2}-i\alpha_S\right)}{\Gamma\left(\frac{1}{2}+i\alpha_S\right)\Gamma\left(\frac{1}{2}-i\alpha_S\right)}$$

$\boxed{T \otimes H^{D-1}}$

$$V(\phi) = \frac{\xi_0}{2}R\phi^2 - \frac{\mu_0^2}{2}\phi^2 + \frac{\lambda_0}{4!}\phi^4 + \frac{\hbar\lambda_0}{4\beta}\frac{a^{3-D}}{(4\pi)^{(D-1)/2}}\Gamma\left(\frac{3-D}{2}\right) \times \int_0^{\phi^2} dm^2 \sum_{n=-\infty}^{\infty} \frac{\Gamma\left(\frac{D-2}{2}+\alpha_H\right)}{\Gamma\left(\frac{4-D}{2}+\alpha_H\right)}$$

T. Hattori, M. Hayashi, T. Inagaki, and Y.Kitadono, hep-th/0408220

- ## Energy momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{g^{\mu\nu}}$$

$$= -\partial_{\mu}\phi\partial_{\nu}\phi + \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_{\rho}\phi\partial_{\sigma}\phi - \left(\frac{1}{2}\mu_0^2 g_{\mu\nu} + \xi_0 \underline{G_{\mu\nu}}\right)\phi^2 + \frac{\lambda_0}{4!}g_{\mu\nu}\phi^4$$

Einstein tensor

- ## Back reaction

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle_E$$

Einstein Eq.

$$\left\langle T_{\mu\nu}\right\rangle_E = \left\langle \underline{T_{\mu\nu}}\right\rangle - \underline{\left\langle T_{\mu\nu}\right\rangle}_0$$

$\left\langle T_{\mu\nu}\right\rangle_E$:renormalized expectation value of energy-momentum tensor

$\left\langle T_{\mu\nu}\right\rangle_0$:expectation value of energy-momentum tensor
in flat space at 0 temperature

…expectation value of the field $\langle\phi\rangle$ is evaluated from Gap eq. :

$$\left.\frac{\delta V(\phi)}{\delta\phi}\right|_{\phi\to\langle\phi\rangle} = 0$$

◆ Energy component of Einstein equation

$$\boxed{-\frac{1}{2}R = 8\pi\left\langle T_{44}\right\rangle_E}$$

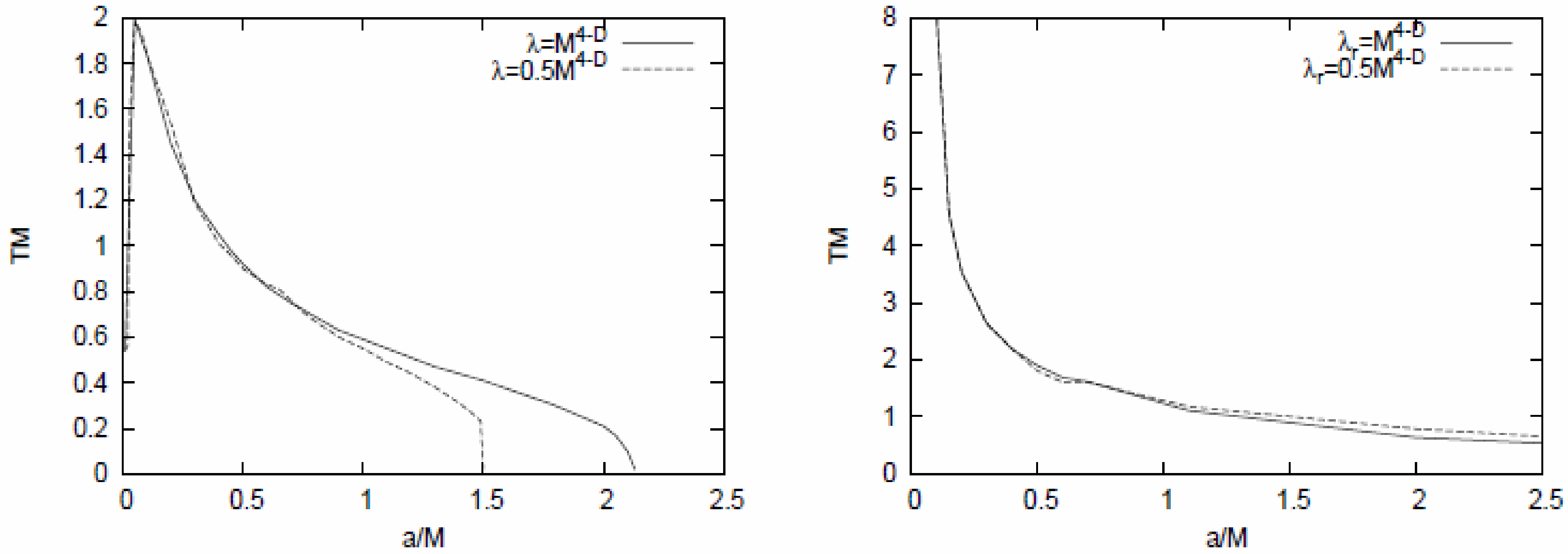
- $T \otimes S^{D-1}$

$$\langle T_{44}\rangle = \frac{1}{2}(-\mu_0^2 + \xi_0 R)\langle\phi\rangle^2 + \frac{\lambda_0}{4!}\langle\phi\rangle^2 - \frac{1}{(4\pi)^{\frac{D-1}{2}}}\Gamma\left(\frac{3-D}{2}\right)Ta^{3-D}$$
$$\times \sum_{n=-\infty}^{n=\infty} \omega_n^2 \frac{\Gamma\left(\frac{D-2}{2} + i\alpha_S\right)\Gamma\left(\frac{D-2}{2} - i\alpha_S\right)}{\Gamma\left(\frac{1}{2} + i\alpha_S\right)\Gamma\left(\frac{1}{2} - i\alpha_S\right)}$$

- $T \otimes H^{D-1}$

$$\langle T_{44}\rangle = \frac{1}{2}(-\mu_0^2 + \xi_0 R)\langle\phi\rangle^2 + \frac{\lambda_0}{4!}\langle\phi\rangle^2 - \frac{1}{(4\pi)^{\frac{D-1}{2}}}\Gamma\left(\frac{3-D}{2}\right)Ta^{3-D}$$
$$\times \sum_{n=-\infty}^{n=\infty} \omega_n^2 \frac{\Gamma\left(\frac{D-2}{2} + \alpha_H\right)}{\Gamma\left(\frac{4-D}{2} + \alpha_H\right)}$$

4. Numerical Calculation



(a) $R \otimes S^{D-1}$, $\mu^2 = 0.1M^2$, $\lambda = M^{4-D}$,$0.5M^{4-D}$

(b) $R \otimes H^{D-1}$,$\mu^2 = 0.1M^2$, $\lambda = M^{4-D}$,$0.5M^{4-D}$

Figure 1. The $T - a$ relationship for a conformally coupled scalar field $\xi = (D-2)/(4D-4)$.

5. Summary

1. Einstein equation for a massive scalar field is exactly solved
   in static and higherly symmetric curved space at finite temperature.
2. The relationship between the temperature and the scale factor is shown as the coupling constant varies.
3. We investigate the four-fermion interaction model now.