Inflationary Perturbations

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Summer Institute 2006
August 23-30
APCTP, Pohang

Outline

Cosmological Perturbations (KAIST lecture notes)

Single Field Inflation

Scale-Invariant Inflationary Perturbations (KAIST lecture notes)

General Slow-Roll (PRD 65 101301; 103508; JCAP 0407012; 0504012)

Multiple Field Inflation

$\delta\mathcal{N}$ (M. Sasaki, E. D. Stewart, Prog. Theor. Phys. 95, 71-78)

δN (H.-C. Lee, M. Sasaki, E. D. Stewart, T. Tanaka, S. Yokoyama, JCAP 0510004)

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Perturbation equations

The evolution equation for

$$\varphi \equiv a\left(\delta\phi - \frac{\dot{\phi}}{H}\mathcal{R}\right)$$

is

$$\frac{d^2\varphi}{d\xi^2} + k^2\varphi - \left[\left(\frac{H}{a\dot{\phi}}\right)\frac{d^2}{d\xi^2}\left(\frac{a\dot{\phi}}{H}\right)\right]\varphi = 0$$

where ξ is minus the conformal time

$$\xi \equiv -\eta = \int_t^{\infty}\frac{dt}{a} = \frac{1}{aH}\left[1 + \mathcal{O}\left(\frac{\dot{H}}{H^2}\right)\right]$$

Defining

$$f(\ln\xi) \equiv \frac{2\pi a\xi\dot{\phi}}{H} \simeq \frac{2\pi\dot{\phi}}{H^2}$$

we can rearrange the equation as

$$\frac{d^2\varphi}{d\xi^2} + k^2\varphi - \frac{2}{\xi^2}\varphi = \left(\frac{f'' - 3f'}{f}\right)\frac{1}{\xi^2}\varphi$$

where

$$f' \equiv \frac{df}{d\ln\xi} \simeq -\frac{1}{H}\frac{df}{dt}$$

General slow-roll approximation

Our equation is

$$\frac{d^2\varphi}{d\xi^2} + k^2\varphi - \frac{2}{\xi^2}\varphi = \left(\frac{f'' - 3f'}{f}\right)\frac{1}{\xi^2}\varphi$$

The homogeneous equation

$$\frac{d^2\varphi_0}{d\xi^2} + k^2\varphi_0 - \frac{2}{\xi^2}\varphi_0 = 0$$

has solution

$$\varphi_0 = \frac{1}{\sqrt{2k}}\left(1 + \frac{i}{k\xi}\right)e^{ik\xi}$$

which gives the scale invariant spectrum

$$\mathcal{P} \simeq \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 \simeq \frac{1}{f^2}$$

we derived previously. Thus we see that

$$\frac{f'}{f}, \frac{f''}{f} \ll 1 \quad \leftrightarrow \quad \text{scale invariance}$$

However, unlike in standard slow-roll, we will not make any further assumptions

$$\frac{f''}{f} \not\ll \frac{f'}{f} \quad \leftrightarrow \quad \frac{dn}{d\ln k} \not\ll n - 1$$

General slow-roll formula

Solving

$$\frac{d^2\varphi}{d\xi^2} + k^2\varphi - \frac{2}{\xi^2}\varphi = \left(\frac{f'' - 3f'}{f}\right)\frac{1}{\xi^2}\varphi$$

perturbatively, and taking the late time limit $\xi \to 0$, we get

$$\ln \mathcal{P}(k) = \int_0^\infty \frac{d\xi}{\xi}\left[-k\xi\, W'(k\xi)\right]\left(\ln\frac{1}{f^2} + \frac{2}{3}\frac{f'}{f}\right)$$

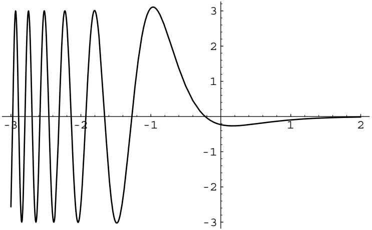

Figure: The window function $-\frac{k}{aH}\, W'\left(\frac{k}{aH}\right)$ as a function of $\ln\frac{aH}{k}$.

General slow-roll formula

Solving

$$\frac{d^2\varphi}{d\xi^2} + k^2\varphi - \frac{2}{\xi^2}\varphi = \left(\frac{f'' - 3f'}{f}\right)\frac{1}{\xi^2}\varphi$$

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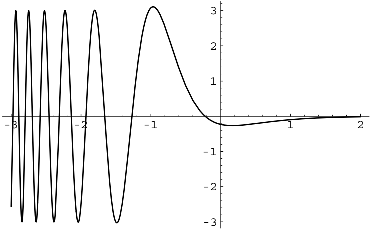

Figure: The window function $-\frac{k}{aH}\, W'\left(\frac{k}{aH}\right)$ as a function of $\ln\frac{aH}{k}$.

Window function $-\frac{k}{aH} W'\left(\frac{k}{aH}\right)$

The window function has the window property

$$\int_0^\infty \frac{dx}{x} \left[-x\, W'(x)\right] = 1$$

and is generated from

$$W(x) = \frac{3\sin(2x)}{2x^3} - \frac{3\cos(2x)}{x^2} - \frac{3\sin(2x)}{2x} - 1$$

which has the asymptotic behavior

$$\lim_{x\to 0} W(x) = \frac{2}{5}x^2 + \mathcal{O}(x^4)$$

Inverse

The general slow-roll formula for the spectrum

$$\ln \mathcal{P}(k) = \int_0^\infty \frac{d\xi}{\xi} \left[-k\xi\, W'(k\xi)\right] \left(\ln \frac{1}{f^2} + \frac{2}{3}\frac{f'}{f}\right)$$

where

$$\frac{1}{f(\ln \xi)} \equiv \frac{H}{2\pi a \xi \dot{\phi}} \simeq \frac{H^2}{2\pi \dot{\phi}}$$

can be inverted to give

$$\ln \frac{1}{f^2} = \int_0^\infty \frac{dk}{k}\, m(k\xi)\, \ln \mathcal{P}$$

where the inverse window function

$$m(x) = \frac{2}{\pi}\left[\frac{1}{x} - \frac{\cos(2x)}{x} - \sin(2x)\right]$$

has the window property

$$\int_0^\infty \frac{dx}{x}\, m(x) = 1$$

and the asymptotic behavior

$$\lim_{x \to 0} m(x) = \frac{4}{3\pi} x^3 + \mathcal{O}(x^5)$$

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Homogeneous background

The background metric is

$$ds^2 = dt^2 - a(t)^2 \delta_{ij}\, dx^i dx^j$$

We define the *e*-folding number as

$$N(t_{\rm fin}, t) \equiv \int_{t_{\rm fin}}^{t} H\, dt$$

where $H \equiv \dot{a}/a$ and $t_{\rm fin}$ is a late time when all trajectories have converged, i.e. a time after complete reheating when the curvature perturbation $\mathcal{R}_{\rm c}$ has become constant.

$\mathcal{N}$

We write the scalar part of the perturbed metric as

$$ds^2 = (1+2A)dt^2 - 2\partial_i B\, dt\, dx^i - a^2\left[(1+2\mathcal{R})\delta_{ij} + 2a^{-2}\partial_i\partial_j E\right] dx^i dx^j$$

The perturbed e-folding number is defined by

$$\mathcal{N}(t_{\rm fin}, t) \equiv \int_{t_{\rm fin}}^{t} \frac{1}{3}\theta\, d\tau$$

where τ is the proper time, $d\tau = (1+A)\, dt$, and θ is the volume expansion rate of the constant time hypersurfaces

$$\frac{1}{3}\theta = H\left(1 + \frac{1}{H}\dot{\mathcal{R}} - A - \frac{q^2}{3H}\mathcal{S}\right)$$

$q^2 = k^2/a^2$ is the physical wave number and

$$\mathcal{S} = \dot{E} - 2HE - B$$

is the shear of the unit vector normal to the constant time hypersurfaces. Therefore

$$\begin{aligned}\mathcal{N}(t_{\rm fin}, t) &= \int_{t_{\rm fin}}^{t} H\left(1 + \frac{1}{H}\dot{\mathcal{R}} - A - \frac{q^2}{3H}\mathcal{S}\right)(1+A)\, dt \\ &= N(t_{\rm fin}, t) + \mathcal{R}(t) - \mathcal{R}(t_{\rm fin}) - \frac{1}{3}\int_{t_{\rm fin}}^{t} q^2\mathcal{S}\, dt\end{aligned}$$

$\delta\mathcal{N}$

$$\begin{aligned}\delta\mathcal{N}(t_{\rm fin},t) &\equiv \mathcal{N}(t_{\rm fin},t) - N(t_{\rm fin},t) \\ &= \mathcal{R}(t) - \mathcal{R}(t_{\rm fin}) - \frac{1}{3}\int_{t_{\rm fin}}^{t} q^2\mathcal{S}\,dt\end{aligned}$$

Taking the initial hypersurface to be flat and the final hypersurface to be comoving

$$\delta\mathcal{N}(t_{\rm fin},t_{\rm ini}) = -\mathcal{R}_{\rm c}(t_{\rm fin}) - \frac{1}{3}\int_{t_{\rm fin}}^{t_{\rm ini}} q^2\mathcal{S}\,dt$$

The Einstein equation gives

$$\dot{\mathcal{S}} + H\mathcal{S} = A + \mathcal{R} + \pi$$

where the matter anisotropic stress π is usually negligible. Thus $q^2\mathcal{S}$ decays rapidly outside the horizon, and this decaying behavior is independent of the choice of gauge. So, taking the initial time $t_{\rm ini}$ sufficiently late, so that scales are sufficiently greater than the horizon, we get our central result

$$\mathcal{R}_{\rm c}(t_{\rm fin}) \simeq -\delta\mathcal{N}(t_{\rm fin},t_{\rm ini})$$

This relation is valid irrespective of whether the background universe is dominated by scalar field or not.

$\delta\mathcal{N}$

$$
\begin{aligned}
\delta\mathcal{N}(t_{\rm fin},t) &\equiv \mathcal{N}(t_{\rm fin},t) - N(t_{\rm fin},t) \\
&= \mathcal{R}(t) - \mathcal{R}(t_{\rm fin}) - \frac{1}{3}\int_{t_{\rm fin}}^{t} q^2\mathcal{S}\,dt
\end{aligned}
$$

Taking the initial hypersurface to be flat and the final hypersurface to be comoving

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\delta\mathcal{N}(t_{\rm fin},t_{\rm ini}) = -\mathcal{R}_{\rm c}(t_{\rm fin}) - \frac{1}{3}\int_{t_{\rm fin}}^{t_{\rm ini}} q^2\mathcal{S}\,dt
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Scalar fields during inflation

We assume that for $t \leq t_{\rm ini}$, i.e. while modes are leaving the horizon during inflation, we can take the action to be

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}R + \frac{1}{2}h_{\mathbf{ab}}g^{\mu\nu}\partial_\mu\phi^{\mathbf{a}}\partial_\nu\phi^{\mathbf{b}} - V(\phi)\right]$$

and the scalar field perturbations on flat hypersurfaces satisfy

$$\delta\ddot{\phi}^{\mathbf{a}}_{\rm f} + 3H\delta\dot{\phi}^{\mathbf{a}}_{\rm f} - R^{\mathbf{a}}{}_{\mathbf{bcd}}\dot{\phi}^{\mathbf{b}}\dot{\phi}^{\mathbf{c}}\delta\phi^{\mathbf{d}}_{\rm f} + q^2\delta\phi^{\mathbf{a}}_{\rm f} + h^{\mathbf{ab}}V_{\phi^{\mathbf{b}}\phi^{\mathbf{c}}}\delta\phi^{\mathbf{c}}_{\rm f} = \frac{1}{a^3}\frac{D}{dt}\left(\frac{a^3\dot{\phi}^{\mathbf{a}}\dot{\phi}^{\mathbf{b}}}{H}\right)h_{\mathbf{bc}}\delta\phi^{\mathbf{c}}_{\rm f}$$

We do not require this effective description to continue to be valid for $t > t_{\rm ini}$. Indeed, we expect that in most cases it will break down some time before $t_{\rm fin}$.

N

For $t \le t_{\rm ini}$, we represent N in phase space $(\phi, \dot\phi)$ as

$$N(t_{\rm fin}; \phi, \dot\phi) \equiv \int_{t_{\rm fin}}^{t(\phi,\dot\phi)} H\, dt$$

where the integral is performed along the trajectory that passes through $(\phi, \dot\phi)$.
The evolution equations for $N_{\dot\phi^{\mathbf{b}}}$ and $N_{\phi^{\mathbf{b}}}$ are

$$\frac{D^2}{dt^2}\left(\frac{h^{\mathbf{ab}} N_{\dot\phi^{\mathbf{b}}}}{a^3}\right) + 3H\frac{D}{dt}\left(\frac{h^{\mathbf{ab}} N_{\dot\phi^{\mathbf{b}}}}{a^3}\right) - R^{\mathbf{a}}{}_{\mathbf{bcd}}\dot\phi^{\mathbf{b}}\dot\phi^{\mathbf{c}}\left(\frac{h^{\mathbf{de}} N_{\dot\phi^{\mathbf{e}}}}{a^3}\right) + h^{\mathbf{ab}} V_{\phi^{\mathbf{b}}\phi^{\mathbf{c}}}\left(\frac{h^{\mathbf{cd}} N_{\dot\phi^{\mathbf{d}}}}{a^3}\right)$$
$$= \frac{1}{a^3}\frac{D}{dt}\left(\frac{a^3\dot\phi^{\mathbf{a}}\dot\phi^{\mathbf{b}}}{H}\right) h_{\mathbf{bc}}\left(\frac{h^{\mathbf{cd}} N_{\dot\phi^{\mathbf{d}}}}{a^3}\right) + \frac{1}{3a^3}\frac{D}{dt}\left(\frac{\dot\phi^{\mathbf{a}}}{H}\right)$$

and

$$\frac{h^{\mathbf{ab}} N_{\phi^{\mathbf{b}}}}{a^3} = -\frac{D}{dt}\left(\frac{h^{\mathbf{ab}} N_{\dot\phi^{\mathbf{b}}}}{a^3}\right) + \frac{N_{\dot\phi^{\mathbf{b}}}\dot\phi^{\mathbf{b}}\dot\phi^{\mathbf{a}}}{2Ha^3} + \frac{\dot\phi^{\mathbf{a}}}{6Ha^3}$$

They have the particular solution

$$\frac{h^{\mathbf{ab}} N_{\dot\phi^{\mathbf{b}}}}{a^3} = \frac{\dot\phi^{\mathbf{a}}}{6H}\int \frac{dt}{a^3}$$

$\delta\phi^{\mathbf{a}}_{\rm f}$

The scalar field perturbations on flat hypersurfaces satisfy

$$\delta\ddot{\phi}^{\mathbf{a}}_{\rm f} + 3H\delta\dot{\phi}^{\mathbf{a}}_{\rm f} - R^{\mathbf{a}}{}_{\mathbf{bcd}}\dot{\phi}^{\mathbf{b}}\dot{\phi}^{\mathbf{c}}\delta\phi^{\mathbf{d}}_{\rm f} + q^2\delta\phi^{\mathbf{a}}_{\rm f} + h^{\mathbf{ab}}V_{\phi^{\mathbf{b}}\phi^{\mathbf{c}}}\delta\phi^{\mathbf{c}}_{\rm f} = \frac{1}{a^3}\frac{D}{dt}\left(\frac{a^3\dot{\phi}^{\mathbf{a}}\dot{\phi}^{\mathbf{b}}}{H}\right)h_{\mathbf{bc}}\delta\phi^{\mathbf{c}}_{\rm f}$$

In the limit $q^2 \longrightarrow 0$, this has the solution

$$\delta\phi^{\mathbf{a}}_{\rm f} \propto \frac{\dot{\phi}^{\mathbf{a}}}{H}$$

corresponding to the super-horizon adiabatic growing mode. Comparing with $N_{\dot{\phi}^{\mathbf{a}}}$ we see that, in the limit $q^2 \longrightarrow 0$, it also has the solution

$$\delta\phi^{\mathbf{a}}_{\rm f} \propto \frac{h^{\mathbf{ab}}N_{\dot{\phi}^{\mathbf{b}}}}{a^3} - \frac{\dot{\phi}^{\mathbf{a}}}{6H}\int_{t_{\rm fin}}^{t}\frac{dt}{a^3}$$

and

$$\delta\dot{\phi}^{\mathbf{a}}_{\rm f} \propto -\frac{h^{\mathbf{ab}}N_{\phi^{\mathbf{b}}}}{a^3} + \frac{N_{\dot{\phi}^{\mathbf{b}}}\dot{\phi}^{\mathbf{b}}\dot{\phi}^{\mathbf{a}}}{2Ha^3} - \frac{1}{6}\frac{D}{dt}\left(\frac{\dot{\phi}^{\mathbf{a}}}{H}\right)\int_{t_{\rm fin}}^{t}\frac{dt}{a^3}$$

corresponding to a super-horizon decaying mode.

δN

We define

$$\begin{aligned}\delta N &\equiv \mathbb{N}_{\phi^{\mathbf{a}}}\delta\phi^{\mathbf{a}} + \mathbb{N}_{\dot\phi^{\mathbf{a}}}\delta\dot\phi^{\mathbf{a}} \\ &\equiv N_{\phi^{\mathbf{a}}}\delta\phi^{\mathbf{a}} + N_{\dot\phi^{\mathbf{a}}}\left(\delta\dot\phi^{\mathbf{a}} - \dot\phi^{\mathbf{a}}A\right) - \frac{q^2}{9H}X\mathcal{S}_{\mathrm{f}}\end{aligned}$$

with X being some, as yet unspecified, function of t. Note that we may interpret

$$\delta\dot\phi^{\mathbf{a}} - \dot\phi^{\mathbf{a}}A = \delta\left(\frac{d\phi^{\mathbf{a}}}{d\tau}\right)$$

$\mathbb{N}_{\phi^{\mathbf{a}}}$ and $\mathbb{N}_{\dot\phi^{\mathbf{a}}}$ are given explicitly by

$$\mathbb{N}_{\phi^{\mathbf{a}}} = N_{\phi^{\mathbf{a}}} - \frac{1}{2H}N_{\dot\phi^{\mathbf{b}}}\dot\phi^{\mathbf{b}}\dot\phi_{\mathbf{a}} - \frac{X}{18H}\frac{D}{dt}\left(\frac{\dot\phi_{\mathbf{a}}}{H}\right)$$

and

$$\mathbb{N}_{\dot\phi^{\mathbf{a}}} = N_{\dot\phi^{\mathbf{a}}} + \frac{X}{18H}\frac{\dot\phi_{\mathbf{a}}}{H}$$

$\delta N_{\rm f}$

Choosing flat hypersurfaces, we have

$$\delta N_{\rm f} = \delta N - \mathcal{R}$$

Using the gauge transformation property of δN, one can readily show that this is a gauge-invariant quantity. Taking the time derivative of $\delta N_{\rm f}$, we find

$$\dot{\delta N}_{\rm f} = -q^2 \mathbb{N}_{\dot{\phi}^{\bf a}} \delta\phi^{\bf a}_{\rm f} - \frac{1}{3} q^2 \left[1 - \left(1 - \frac{\epsilon}{3} \right) X + \frac{1}{3H} \dot{X} \right] \mathcal{S}_{\rm f}$$

where $\epsilon \equiv -\dot{H}/H^2$. Thus $\delta N_{\rm f}$ is constant on super-horizon scales.
In the special case that our scalar field description is still valid at $t = t_{\rm fin}$, i.e. if it is valid beyond the point where convergence of trajectories occurs, we can evaluate the constant by inserting the super-horizon adiabatic growing mode into the definition of δN to get

$$\delta N_{\rm f}(t_{\rm ini}) = \delta N_{\rm f}(t_{\rm fin}) = \left. \frac{H \dot{\phi}_{\bf a} \delta\phi^{\bf a}_{\rm f}}{\dot{\phi}_{\bf b} \dot{\phi}^{\bf b}} \right|_{t=t_{\rm fin}} = -\mathcal{R}_{\rm c}(t_{\rm fin})$$

Thus we have established the equivalence of $\delta N_{\rm f}$ and $\delta \mathcal{N}$ on super-horizon scales in this special case, and the natural form of $\delta N_{\rm f}$ convinces us that this is true generally.

δN_{f}

Choosing flat hypersurfaces, we have

$$\delta N_{\mathrm{f}} = \delta N - \mathcal{R}$$

Using the gauge transformation property of δN, one can readily show that this is a gauge-invariant quantity. Taking the time derivative of δN_{f}, we find

$$\delta\dot{N}_{\mathrm{f}} = -q^2 \mathbb{N}_{\dot{\phi}^{\mathbf{a}}} \delta\phi^{\mathbf{a}}_{\mathrm{f}} - \frac{1}{3} q^2 \left[1 - \left(1 - \frac{\epsilon}{3} \right) X + \frac{1}{3H} \dot{X} \right] \mathcal{S}_{\mathrm{f}}$$

where $\epsilon \equiv -\dot{H}/H^2$. Thus δN_{f} is constant on super-horizon scales.
Taking the derivative again we get the equation of motion for δN_{f}

$$\begin{aligned} \delta\ddot{N}_{\mathrm{f}} + 5H\,\delta\dot{N}_{\mathrm{f}} + q^2 \delta N_{\mathrm{f}} &= -2q^2 \dot{\mathbb{N}}_{\dot{\phi}^{\mathbf{a}}} \delta\phi^{\mathbf{a}}_{\mathrm{f}} \\ &\quad - \frac{2}{3} H q^2 \left[1 - \left(1 - \frac{\epsilon}{3} - \frac{\dot{\epsilon}}{6H} \right) X - \frac{1 - 2\epsilon}{6H} \dot{X} + \frac{1}{6H^2} \ddot{X} \right] \mathcal{S}_{\mathrm{f}} \end{aligned}$$

X

We have

$$\delta\dot{N}_{\mathrm{f}} = -q^2 \mathbb{N}_{\dot{\phi}^{\mathbf{a}}} \delta\phi^{\mathbf{a}}_{\mathrm{f}} - \frac{1}{3} q^2 \left[1 - \left(1 - \frac{\epsilon}{3} \right) X + \frac{1}{3H} \dot{X} \right] \mathcal{S}_{\mathrm{f}}$$

and

$$\delta\ddot{N}_{\mathrm{f}} + 5H\,\delta\dot{N}_{\mathrm{f}} + q^2 \delta N_{\mathrm{f}} = -2q^2 \dot{\mathbb{N}}_{\dot{\phi}^{\mathbf{a}}} \delta\phi^{\mathbf{a}}_{\mathrm{f}}$$
$$- \frac{2}{3} H q^2 \left[1 - \left(1 - \frac{\epsilon}{3} - \frac{\dot{\epsilon}}{6H} \right) X - \frac{1 - 2\epsilon}{6H} \dot{X} + \frac{1}{6H^2} \ddot{X} \right] \mathcal{S}_{\mathrm{f}}$$

Choosing

$$X = -3H \left(a^3 \int_{t_{\mathrm{fin}}}^{t} \frac{dt}{a^3} \right) = 1 + \mathcal{O}\left(\frac{\dot{H}}{H^2} \right)$$

in which case

$$\mathbb{N}_{\dot{\phi}^{\mathbf{a}}} = N_{\dot{\phi}^{\mathbf{a}}} - \frac{1}{6} \left(a^3 \int_{t_{\mathrm{fin}}}^{t} \frac{dt}{a^3} \right) h_{\mathbf{ab}} \frac{\dot{\phi}^{\mathbf{b}}}{H} \quad , \quad \frac{\mathbb{N}_{\phi^{\mathbf{a}}}}{a^3} = -\frac{D}{dt} \left(\frac{\mathbb{N}_{\dot{\phi}^{\mathbf{a}}}}{a^3} \right)$$

and

$$\delta N = N_{\phi^{\mathbf{a}}} \delta\phi^{\mathbf{a}} + N_{\dot{\phi}^{\mathbf{a}}} \left(\delta\dot{\phi}^{\mathbf{a}} - \dot{\phi}^{\mathbf{a}} A \right) + \frac{1}{3} q^2 \left(a^3 \int_{t_{\mathrm{fin}}}^{t} \frac{dt}{a^3} \right) \mathcal{S}_{\mathrm{f}}$$

we find the $\mathcal{S}_{\mathrm{f}}$ terms disappear

X

We have

$$\delta\dot{N}_{\rm f} = -q^2 \mathbb{N}_{\dot{\phi}^{\mathbf{a}}} \delta\phi_{\rm f}^{\mathbf{a}}$$

and

$$\delta\ddot{N}_{\rm f} + 5H\,\delta\dot{N}_{\rm f} + q^2 \delta N_{\rm f} = -2q^2 \dot{\mathbb{N}}_{\dot{\phi}^{\mathbf{a}}} \delta\phi_{\rm f}^{\mathbf{a}}$$

Choosing

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in which case

$$\mathbb{N}_{\dot{\phi}^{\mathbf{a}}} = N_{\dot{\phi}^{\mathbf{a}}} - \frac{1}{6}\left(a^3 \int_{t_{\rm fin}}^{t} \frac{dt}{a^3}\right) h_{\mathbf{ab}} \frac{\dot{\phi}^{\mathbf{b}}}{H} \quad , \quad \frac{\mathbb{N}_{\phi^{\mathbf{a}}}}{a^3} = -\frac{D}{dt}\left(\frac{\mathbb{N}_{\dot{\phi}^{\mathbf{a}}}}{a^3}\right)$$

and

$$\delta N = N_{\phi^{\mathbf{a}}} \delta\phi^{\mathbf{a}} + N_{\dot{\phi}^{\mathbf{a}}}\left(\delta\dot{\phi}^{\mathbf{a}} - \dot{\phi}^{\mathbf{a}} A\right) + \frac{1}{3} q^2 \left(a^3 \int_{t_{\rm fin}}^{t} \frac{dt}{a^3}\right) \mathcal{S}_{\rm f}$$

we find the $\mathcal{S}_{\rm f}$ terms disappear

$\delta\phi_{\parallel}^{\mathbf{a}}$ and $\delta\phi_{\perp}^{\mathbf{a}}$

Our equations are

$$\delta\dot{N}_{\rm f} = -q^2 \mathbb{N}_{\dot{\phi}^{\mathbf{a}}} \delta\phi_{\rm f}^{\mathbf{a}}$$

and

$$\delta\ddot{N}_{\rm f} + 5H\,\delta\dot{N}_{\rm f} + q^2 \delta N_{\rm f} = -2q^2 \dot{\mathbb{N}}_{\dot{\phi}^{\mathbf{a}}} \delta\phi_{\rm f}^{\mathbf{a}}$$

We decompose $\delta\phi_{\rm f}^{\mathbf{a}}$ into relevant and irrelevant components

$$\delta\phi_{\rm f}^{\mathbf{a}} = \delta\phi_{\parallel}^{\mathbf{a}} + \delta\phi_{\perp}^{\mathbf{a}}$$

with the relevant direction defined by $\mathbb{N}_{\dot{\phi}^{\mathbf{a}}}$

$$\delta\phi_{\parallel}^{\mathbf{a}} \equiv P_{\mathbf{b}}^{\mathbf{a}}\,\delta\phi_{\rm f}^{\mathbf{b}} \quad , \quad \delta\phi_{\perp}^{\mathbf{a}} \equiv Q_{\mathbf{b}}^{\mathbf{a}}\,\delta\phi_{\rm f}^{\mathbf{b}}$$

where

$$P_{\mathbf{b}}^{\mathbf{a}} \equiv \frac{h^{\mathbf{ac}} \mathbb{N}_{\dot{\phi}^{\mathbf{c}}} \mathbb{N}_{\dot{\phi}^{\mathbf{b}}}}{h^{\mathbf{de}} \mathbb{N}_{\dot{\phi}^{\mathbf{d}}} \mathbb{N}_{\dot{\phi}^{\mathbf{e}}}} \quad , \quad Q_{\mathbf{b}}^{\mathbf{a}} \equiv \delta_{\mathbf{b}}^{\mathbf{a}} - P_{\mathbf{b}}^{\mathbf{a}}$$

so that $\delta\phi_{\parallel}^{\mathbf{a}}$ can be expressed entirely in terms of $\delta N_{\rm f}$. Then

$$\delta\ddot{N}_{\rm f} + 5H\,\delta\dot{N}_{\rm f} + q^2 \delta N_{\rm f} = 2\left(\frac{h^{\mathbf{ab}} \dot{\mathbb{N}}_{\dot{\phi}^{\mathbf{a}}} \mathbb{N}_{\dot{\phi}^{\mathbf{b}}}}{h^{\mathbf{cd}} \mathbb{N}_{\dot{\phi}^{\mathbf{c}}} \mathbb{N}_{\dot{\phi}^{\mathbf{d}}}}\right) \delta\dot{N}_{\rm f} - 2q^2 \dot{\mathbb{N}}_{\dot{\phi}^{\mathbf{a}}} \delta\phi_{\perp}^{\mathbf{a}}$$

General slow-roll equation

Our equation is

$$\delta\ddot{N}_{\rm f} + 5H\,\delta\dot{N}_{\rm f} + q^2\delta N_{\rm f} = 2\left(\frac{h^{\bf ab}\dot{\mathbb{N}}_{\dot\phi^{\bf a}}\mathbb{N}_{\dot\phi^{\bf b}}}{h^{\bf cd}\mathbb{N}_{\dot\phi^{\bf c}}\mathbb{N}_{\dot\phi^{\bf d}}}\right)\delta\dot{N}_{\rm f} - 2q^2\dot{\mathbb{N}}_{\dot\phi^{\bf a}}\delta\phi_\perp^{\bf a}$$

Changing time variable to $x \equiv k\xi \simeq k/aH$, where $\xi = -\int dt/a$ is minus the conformal time, we obtain

$$\frac{d^2\delta N_{\rm f}}{dx^2} - \frac{4}{x}\frac{d\delta N_{\rm f}}{dx} + \delta N_{\rm f} = \frac{2}{x}\frac{\Pi'}{\Pi}\frac{d\delta N_{\rm f}}{dx} - 2\dot{\mathbb{N}}_{\dot\phi^{\bf a}}\delta\phi_\perp^{\bf a}$$

where

$$\Pi^2(\ln\xi) \equiv \left(\frac{3}{2\pi a^2\xi^2}\right)^2 h^{\bf ab}\mathbb{N}_{\dot\phi^{\bf a}}\mathbb{N}_{\dot\phi^{\bf b}} \simeq \left(\frac{3H^2}{2\pi}\right)^2 h^{\bf ab}\mathbb{N}_{\dot\phi^{\bf a}}\mathbb{N}_{\dot\phi^{\bf b}}$$

and

$$\Pi' \equiv \frac{d\Pi}{d\ln\xi}$$

General slow-roll formula

The general slow-roll solution of

$$\delta N_{\mathrm{f}}'' - \frac{4}{x}\,\delta N_{\mathrm{f}}' + \delta N_{\mathrm{f}} = \frac{2}{x}\frac{\Pi'}{\Pi}\,\delta N_{\mathrm{f}}' - 2\dot{\mathbb{N}}_{\dot{\phi}^{\mathbf{a}}}\,\delta\phi_{\perp}^{\mathbf{a}}$$

is

$$\begin{aligned} P(k) \quad = \quad & \Pi^2(\infty) + 2\,\Pi^2(\infty)\int_0^\infty \frac{d\xi}{\xi}\,z_{\parallel}(k\xi)\,\frac{\Pi'}{\Pi}(\xi) \\ & - \frac{2}{\pi}\sum_{i=1}^{D}\Pi_i\int_0^\infty \frac{d\xi}{\xi}\,z_{\perp}(k\xi)\,\dot{\mathbb{N}}_{\dot{\phi}^{\mathbf{a}}}(\xi)\,\mathrm{Im}\left[\delta\phi_{i\perp}^{\mathbf{a}}(k,\xi)\right] \\ & + \frac{1}{\pi^2}\sum_{i=1}^{D}\left|\int_0^\infty \frac{d\xi}{\xi}\,z_{\perp}(k\xi)\,\dot{\mathbb{N}}_{\dot{\phi}^{\mathbf{a}}}(\xi)\,\delta\phi_{i\perp}^{\mathbf{a}}(k,\xi)\right|^2 \end{aligned}$$

where

$$z_{\parallel}(x) = \frac{3\sin(2x)}{x^3} - \frac{6\cos(2x)}{x^2} - \frac{4\sin(2x)}{x} + \cos(2x) - 1$$

and

$$z_{\perp}(x) = \frac{9\sin x}{x^3} - \frac{9\cos x}{x^2} - \frac{3\sin x}{x}$$

If we neglect the $\delta\phi^{\mathbf{a}}_{\perp}$ terms, the formula for the spectrum reduces to

$$\begin{aligned} \ln P(k) &= \ln \Pi^2(\infty) + 2\int_0^\infty \frac{d\xi}{\xi}\, z_{\parallel}(k\xi)\, \frac{\Pi'}{\Pi}(\xi) \\ &= \int_0^\infty \frac{d\xi}{\xi}\left[-k\xi\, W'(k\xi)\right]\left[\ln \Pi^2 + \frac{2}{3}\frac{\Pi'}{\Pi}\right] \end{aligned}$$

which can be inverted to give

$$\ln \Pi^2 = \int_0^\infty \frac{dk}{k}\, s(k\xi) \ln P$$

where

$$s(x) \equiv \frac{2}{\pi}\left[\frac{3}{x^3} - \frac{3\cos(2x)}{x^3} - \frac{6\sin(2x)}{x^2} + \frac{2}{x} + \frac{4\cos(2x)}{x} + \sin(2x)\right]$$

has the window property

$$\int_0^\infty \frac{dx}{x}\, s(x) = 1$$

and the asymptotic behavior

$$\lim_{x\to 0} s(x) = \frac{4}{45\pi}x^5 + \mathcal{O}(x^7)$$

Cosmological Perturbations (KAIST lecture notes)

Single Field Inflation

Scale-Invariant Inflationary Perturbations (KAIST lecture notes)

General Slow-Roll (PRD 65 101301; 103508; JCAP 0407012; 0504012)

Multiple Field Inflation

$\delta\mathcal{N}$ (M. Sasaki, E. D. Stewart, Prog. Theor. Phys. 95, 71-78)

δN (H.-C. Lee, M. Sasaki, E. D. Stewart, T. Tanaka, S. Yokoyama, JCAP 0510004)

Modular Inflation (K. Kadota, E. D. Stewart, JHEP 0307013; JHEP 0312008)