Qubits, quantum control & physical implementation.

- Mini lecture series (8 weeks)
- 2weeks : Qubits, density matrix, decoherence, quantum control (from NMR)
- 2weeks : semiconductor mesoscopic physics, Quantum dot basics
- 3weeks : spin qubits in semiconductor quantum dot (Loss-DiVincenzo, Singlet-Triplet, Exchange-only ...)
- 1 week : Recent developments & works in SNU lab.

◦ Lecture 1. Qubits & Coherence

- Qubit : Quantum two level system. In general, quantum superposition of two energy eigenstates of a physical system energetically well separated from other states.

(e.g. spin-$\frac{1}{2}$: canonical two level system)
under external field

say, $|\uparrow\rangle \equiv |0\rangle$, $|\downarrow\rangle \equiv |1\rangle$, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

α & β are in general complex # $\rightarrow$ $|\alpha|^2 + |\beta|^2 = 1$

General superposition state of a qubit

- When you measure a qubit,

$|\alpha|^2 \Rightarrow P_0$: probability to find $|0\rangle$ state

$|\beta|^2 \Rightarrow P_1$: " " " $|1\rangle$ state

- Note (important!)

- Consider a partiqular qubit state

$$|+\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \leftarrow P_0 = \frac{1}{2} = P_1$$

- And, just a classical bit fluctuating in time

→ What is the crucial difference? $\boxed{\text{Q \& A}}$

- And, how do you experimentally distinguish?

$\boxed{\text{Q \& A}}$

- Coherence: A system's capability to exhibit 'interference'

- Qubit: phase coherent, Fluctuating bit: Incoherent (statistical Mixture)

- Qubit, if perfectly isolated from environment, is kept coherent indefinately. But this is never true in reality. Qubit always interact with environment, including the experimenter who contols and measures the qubit. Thus,

$$\text{Qubit} \xrightarrow[\text{in time}]{} \text{Classical (Fluctuating) bit}$$

- Now, can we think of a two level state that is in between fully coherent and fully classical?
$\underbrace{\text{fully coherent}}_{\text{quantum}}$

- Mathematical tool for this is called 'Density matrix'

- Definition : Density Matrix
For a set of normalized $\{|\varphi_i\rangle\}$

$$\rho \equiv \sum_i p_i |\varphi_i\rangle\langle\varphi_i| \qquad \left(\begin{matrix}\text{of course} \\ p_i \geq 0 \ , \ \sum_i p_i = 1\end{matrix}\right)$$

ρ ↑ density matrix

p_i ↑ Probability to be in a state $|\varphi_i\rangle$

- With this tool,
  - Fluctuating classical bit (SM)
  $: \rho_{SM} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$
  - a Qubit $|+\rangle$

$$: \rho_{|+\rangle} = 1 \times |+\rangle\langle +| = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$$
$$= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

- In matrix, $\rho_{SM} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $\rho_{|+\rangle} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

- And, as promised, e.g.

70% in $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, 30% in $|-\rangle$
$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$\Rightarrow$ Do your Self.

◦ Properties of Density Matrix

1) ρ: Hermitian
2) $Tr(\rho)=1$
3) $\rho^2=\rho$ iff pure
4) $Tr(\rho^2)\le 1$ $=1$ iff pure

$\underbrace{\text{parity}} \longrightarrow$ 4)

◦ Expectation value: $\langle A\rangle = Tr(\rho A)$

$\Rightarrow$ Again, How to Experimentally distinguish $\begin{cases} \rho_{SM} \\ \rho_{|+\rangle} \end{cases}$?

$\boxed{Q\&A}$, QST: Experimentally reconstruct a density matrix of a quantum state by measuring its all possible basis projections.

◦ DM for a qubit

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & \rho_{22} \end{pmatrix} \quad \rho_{11}+\rho_{22}=1$$

real

Pauli Matrices

$$\rho = \frac{1}{2}\left(\mathbb{1} + \sum m_i \hat{\sigma}_i\right)$$

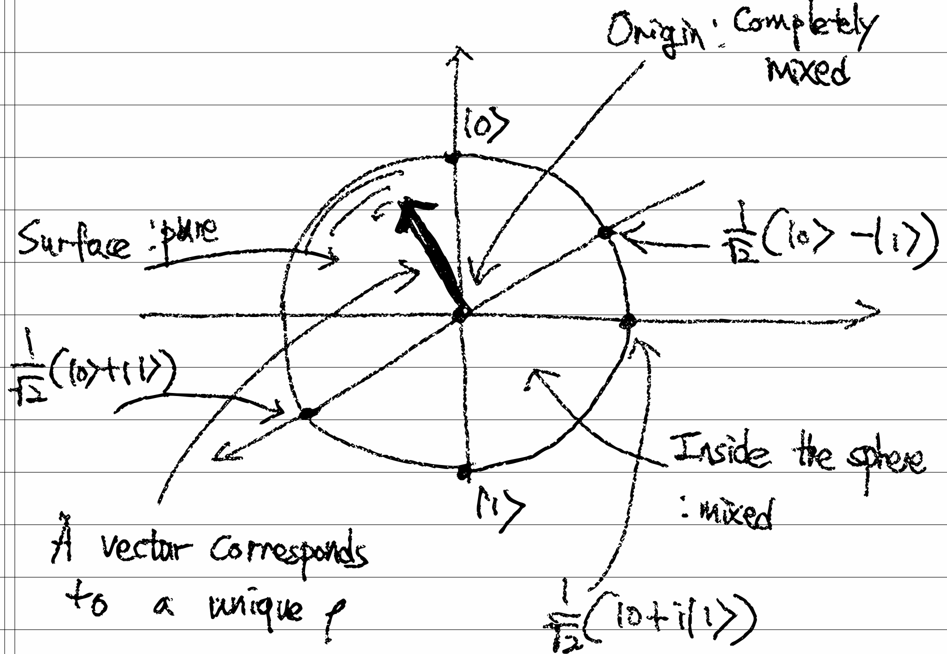
Pauli rep. of ρ for a qubit

◦ Purity in this rep. $Tr(\rho^2) = \frac{1}{2}\left(1+\sum_i m_i^2\right) \le 1$

$\Rightarrow |\vec{m}| \le 1$

$\downarrow$

Very popular Graphical way to represent ρ of a qubit, Motivated by this, B Bloch sphere



- Control of quantum two level system

- Consider one has ability to set initial qubit state $|0\rangle$
- How to control its superposition? : $|0\rangle \rightarrow \underset{?}{\alpha|0\rangle + \beta|1\rangle}$

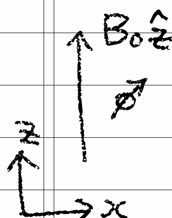
- For our purpose of discussion, two main methods

① Non-adiabatic control
② Resonant control

} Currently widely used in all platforms (SC, IT, DC, QD, ...)
maybe except topological quantum computing
→ Later...

① Non-adiabatic control.

- Any two level system : Think as a spin $\frac{1}{2}$



$$H = \gamma \vec{B}\cdot\vec{S} = \frac{1}{2}\gamma\hbar B_0 \sigma_z = \frac{\hbar\omega_0}{2}\sigma_z$$

Gyromagnetic Ratio (γ) Pauli Matrix (σ_z) Larmor freq. (ω_0)

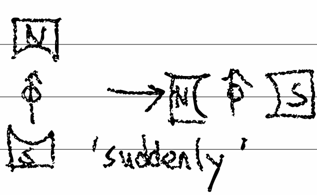
- In other words, $H = \frac{\hbar\omega_0}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

- Eigenstates are $|\uparrow\rangle = |0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|\downarrow\rangle = |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

- Suppose we wait long enough so that we know

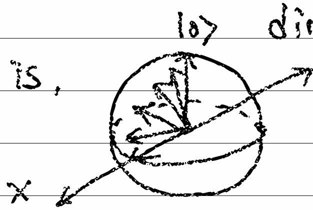
$$|\varphi(0)\rangle = |\uparrow\rangle = |0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

- To control its superposition, 'suddenly' change the direction of the external field to, say, x-direction.



, What happens? $\boxed{\text{Q\&A}}$
How many of you think the spin will align to X direction? $\boxed{\text{Q\&A}}$

- That is,



? → If Slow : Yes
→ If suddenly? : No

- Slow case : explain
- Sudden case :

$$H(t=0) = \frac{\hbar\omega_0}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \rightarrow H(t>0) = \frac{\hbar\omega_0}{2}\overset{\sigma_x}{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}$$

$$|\varphi(0)\rangle = \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix}\right)$$

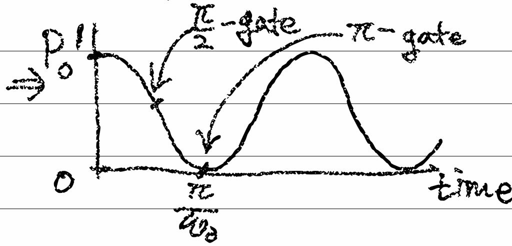
$$|\varphi(t)\rangle = \frac{1}{2}e^{-i\frac{\omega_0}{2}t}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2}e^{+i\frac{\omega_0}{2}t}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \frac{1}{2}\underbrace{\cancel{e^{-i\frac{\omega_0}{2}t}}}_{\substack{\text{무시} \\ \text{(Global phase)}}}\left\{\begin{pmatrix}1\\1\end{pmatrix} + e^{i\omega_0 t}\begin{pmatrix}1\\-1\end{pmatrix}\right\}$$

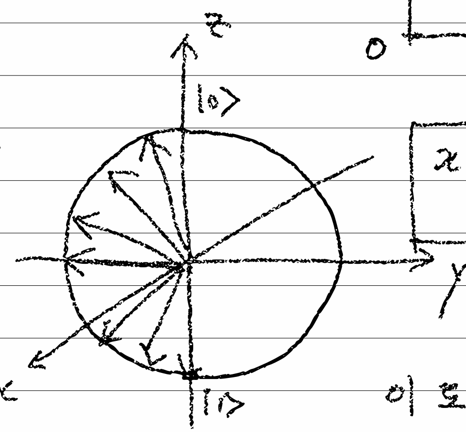
$$P_0(t) = \langle\varphi(0)|\varphi(t)\rangle^2 = \left[\frac{1}{4}\left(2+2e^{i\omega_0 t}\right)\right]^2 = \left[\frac{1}{2}\left(1+e^{i\omega_0 t}\right)\right]^2$$

$$= \frac{1}{4}\left(2+e^{i\omega_0 t}+e^{-i\omega_0 t}\right) = \frac{1}{4}\left(2+2\cos\omega_0 t\right)$$

$$= \frac{1}{2}\left(1+\cos\omega_0 t\right) \Rightarrow$$



⇒



$\boxed{\text{x-axis rotation (precession)}}$

The point :
Field의 방향 (H에 σ_x σ_y σ_z ?) 이 도는 축을 결정

② Resonant control

- Again consider, $\hat{H}_0 = \frac{\hbar \omega_0}{2} \hat{\sigma}_z$

- Apply Harmonic Radiation

$$\hat{H}^{NMR} = \underbrace{\frac{\hbar \omega_0}{2} \sigma_z}_{\hat{H}_0} + \hbar \eta \left(\sigma_x \cos \omega t + \sigma_y \sin \omega t \right)$$

- Schrödinger eq. $i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}^{NMR} |\psi(t)\rangle$

Trick : Use $e^{i\frac{\omega t}{2}\sigma_z} \sigma_x e^{-i\frac{\omega t}{2}\sigma_z} = \cos \omega t \, \sigma_x - \sin \omega t \, \sigma_y$

& $e^{i\frac{\omega t}{2}\sigma_z} \sigma_y e^{-i\frac{\omega t}{2}\sigma_z} = \sin \omega t \, \sigma_x + \cos \omega t \, \sigma_y$

Also, transform $|\psi(t)\rangle = e^{-i\frac{\omega t}{2}\sigma_z} |\varphi(t)\rangle$

- S-E in 'a rotating frame' is $i\hbar \frac{\partial |\varphi(t)\rangle}{\partial t} = H^{NMR}_{rot}(\omega) |\varphi(t)\rangle$

where $H^{NMR}_{rot}(\omega) = \frac{\hbar}{2}(\omega_0 - \omega)\sigma_z + \frac{\hbar \eta}{2} \sigma_x$

$\hat{H}^{NMR} \rightarrow H^{NMR}_{rot}$ (연습)

- Also, time evolution operator $U^{NMR}_{rot}(\omega) = e^{-iH^{NMR}_{rot} t/\hbar}$

$= e^{-i\Omega(\omega)\, \hat{n} \cdot \hat{\sigma}\, t}$

'Rabi frequency'

where $\Omega(\omega) = \sqrt{\left(\frac{(\omega_0 - \omega)}{2}\right)^2 + (\eta)^2}$

$\omega_0 - \omega$: detuning $\equiv \delta$

rotating axis $\hat{n} = \dfrac{\frac{1}{2}(\omega_0-\omega)\hat{z} + \eta\hat{x}}{\sqrt{\left(\frac{\omega_0-\omega}{2}\right)^2 + (\eta)^2}}$

- Resonant case $\omega = \omega_0$, $\Omega(\omega_0) = \eta$ and $\hat{n} = \hat{x}$ in the rotating frame.

- So that $U^{NMR}_{rot}(\omega_0) = e^{-i\sigma_x \eta t/\hbar}$

- To convert back to Lab frame, use

$$U_{Lab} = e^{-i\frac{\omega t}{2}\sigma_z}\, U_{rot}\, e^{i\frac{\omega t}{2}\sigma_z}$$

$\boxed{\text{DIY}}$: Work this derivation out

$\boxed{\text{Q\&A}}$: What if we apply $\sim\cos(\omega t+\phi)$?

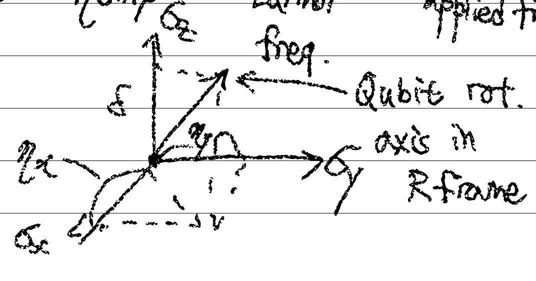
→ With ϕ, we can control rotation axis in XY plane on the Bloch sphere

■ Things to remember

Rotating frame H　　　　　　Lab frame H

$$H_{rot} = \frac{\hbar}{2}\left(\delta\sigma_z + \eta_x\sigma_x + \eta_y\sigma_y\right) \qquad H_{lab} = \frac{\hbar\omega_0}{2}\sigma_z + \hbar\eta\cos(\omega t+\phi)\sigma_x$$

detuning　　$\eta\cos\phi$　　$\eta\sin\phi$　　Larmor freq.　　applied freq.



∘ Show animation (ppt)

∘ DIY : Synthesize various 1Q gates

: $X_{\frac{\pi}{2}}, Y_{\frac{\pi}{2}}, X_{\pi}, Y_{\pi}, H \cdots\cdots, Z$

∘ ⇒ Talk about noise channels : $T_1, T_2^*, T_2 \cdots$

◎ Lecture 2 : 2Q gate, Entanglement & Universal gate set.

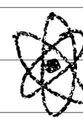
∘ Preliminary : Qubit이 두개이상 있는것은 어떻게 표현?

↳ Composite system을 기술하는 방법?

∘ Hilbert space of composite system.

고상하게 말하자면: The observables for a quantum sys. AB with observable algebra A & B, respectively is $A \otimes B$ ← Tensor product.

Ex) 2 qubits

A   B

$A \simeq M_2(\mathbb{C})$

$B \simeq M_2(\mathbb{C})$

↳ That is, if system A, B is described by 2x2 matrices

$$A \otimes B = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \simeq M_4(\mathbb{C})$$

e.g.)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa' & ab' & \cdot & \cdot \\ ac' & ad' & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

∘ States of composite systems

→ Underlying Hilbert space $\mathcal{H}$ of AB $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

→ Simple states: product states $|\phi_A\rangle \otimes |\phi_B\rangle$ (denoted as $|\phi_A \phi_B\rangle$)

↳ with $\hat{\rho} = |\phi_A\rangle\langle\phi_A| \otimes |\phi_B\rangle\langle\phi_B| = \hat{\rho}_A \otimes \hat{\rho}_B$

∘ $\boxed{\text{Q\&A}}$ Are all states product state? No

• States which aren't products are correlated.

↳ Among correlated states: Entangled states

"The fact that we have more than just product states is why we have decoherence"

∘ Canonical Entangled state: Spin singlet ← One of Bell states

$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ ← cannot be $|\phi_A\rangle \otimes |\phi_B\rangle$

$\rho = \frac{1}{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)$

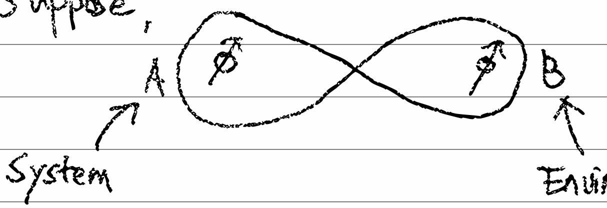
$$\rightarrow \begin{pmatrix} \langle\uparrow\uparrow|\rho|\uparrow\uparrow\rangle & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pure two spin singlet state

(DIY: Check purity $\mathrm{Tr}(\rho^2)$)

- This is a good example to explain decoherence

- Suppose,



We know AB (sys. + env.) as a whole is <u>pure</u>, coherent.

- And suppose that we cannot know the state of B (env.). We can only measure the system (A).

- Mathematically, this corresponds to 'Trace out the env.'
: Average out, Integrate out -- : How to do this?
Take Partial Trace.

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_j \langle \psi_j|_B \, p_i \, |\Psi\rangle_{AB} \langle \Psi|_{AB} \, |\psi_j\rangle_B$$

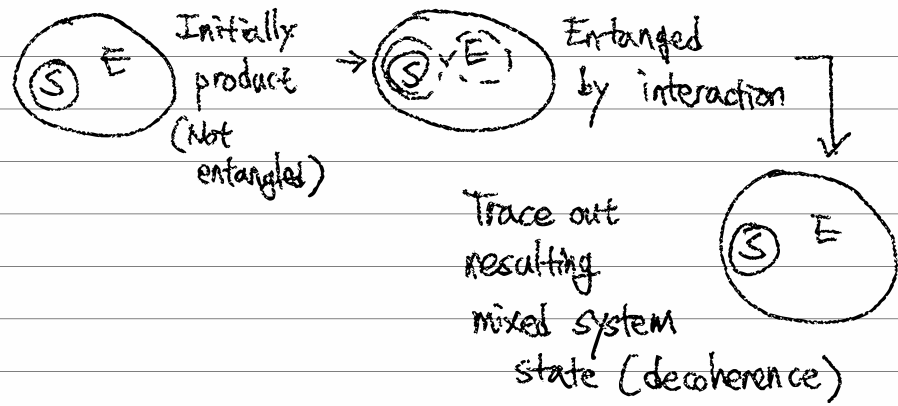
- For above singlet state, $\rho_A = \mathrm{Tr}_B \, \rho_{AB}$

$$= \frac{1}{2} \langle \uparrow|_B \left(\rho_{AB} \right) |\uparrow\rangle_B + \frac{1}{2} \langle \downarrow|_B \left(\rho_{AB} \right) |\downarrow\rangle$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \longleftarrow \text{Now, A is completely } \underline{\text{mixed}}.$$

- If $|\Psi_{AB}\rangle$ is entangled, $\hat{\rho}_A$ and $\hat{\rho}_B$ are mixed. In QM we can have a maximally complete description of joint, yet incomplete knowledge of the part.

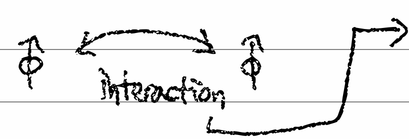
- Basically, this is how we view the process of losing coherence



- More on entanglement: So what exactly is entanglement? - nonlocality
  → Will be discussed more later (Introduce a brief, insightful video clip)

- How to check entanglement? How much entanglement?
  ↪ Also, later.

- Physical situation



$\rightarrow \frac{J_{12}}{2} \sigma_{z_A} \cdot \sigma_{z_B}$

⇒ Show animation (ppt)

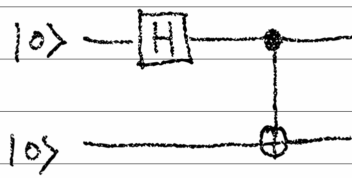
⇒ How to systematically generate this entanglement?
(Two qubit gate)

- Example: CNOT, CPHASE → Invert phase of 'target' " "
  ↪ Flip 'target' qubit iff 'control' is $|1\rangle$

- Many methods exist, but here we focus on CNOT with resonant method.

$\Rightarrow$ Show slide (ppt)

- With this, how to create Bell state?



: $|00\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|0\rangle$

$= \frac{1}{\sqrt{2}}(|00\rangle+|10\rangle) \xrightarrow{\text{CNOT}} \underbrace{\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)}_{\text{Entangled Bell state.}}$

- Including entangled + product states, 1Q, 2Q gate set forms 'Universal gate set', meaning we can approximate any Unitary process with arbitrary precision.

$\Rightarrow$ (More rigorous def. & proof. in QC textbook)

○ Further studies

- Open Q. sys., Master eq. in the Markov approx.

▨ Anyway, this is roughly it for Qubit, and Qubit control ▨