The Shape of Complexity: Geometry, Topology, and Statistics in Complex Systems

Lecture 1: Low-dimensional topology and non-Euclidean geometry in Nature

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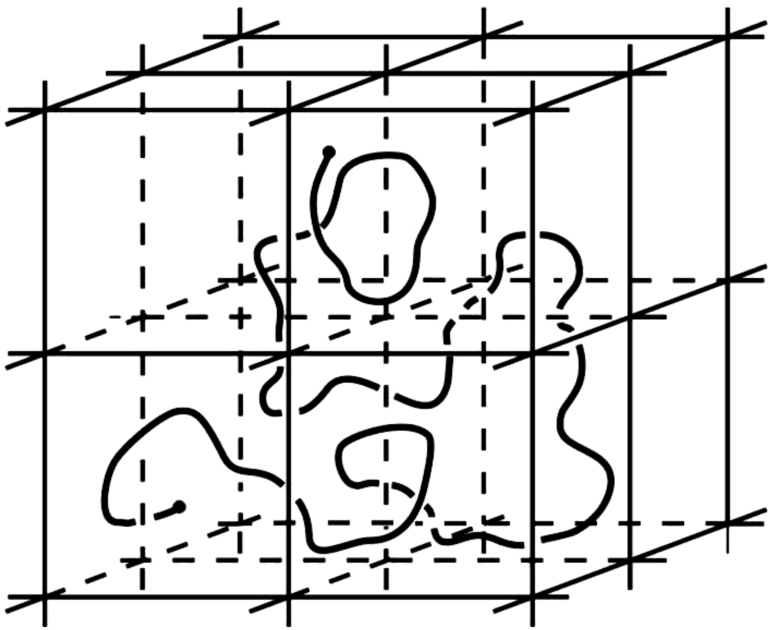
1. Random walks on hyperbolic graphs and surfaces with application to statistics of knots

2. Isometric embedding of an uniform unbound Cayley tree in 3D and a profile of a salad leaf

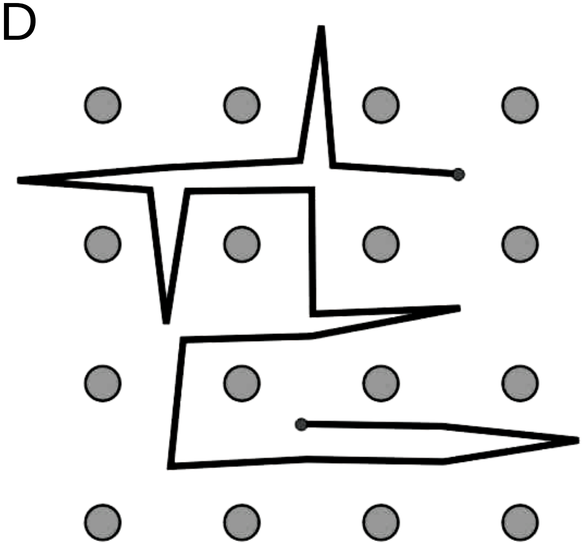
1. Random walks on hyperbolic graphs and surfaces with application to statistics of knots

Polymer entanglement in a lattice of topological obstacles

3D

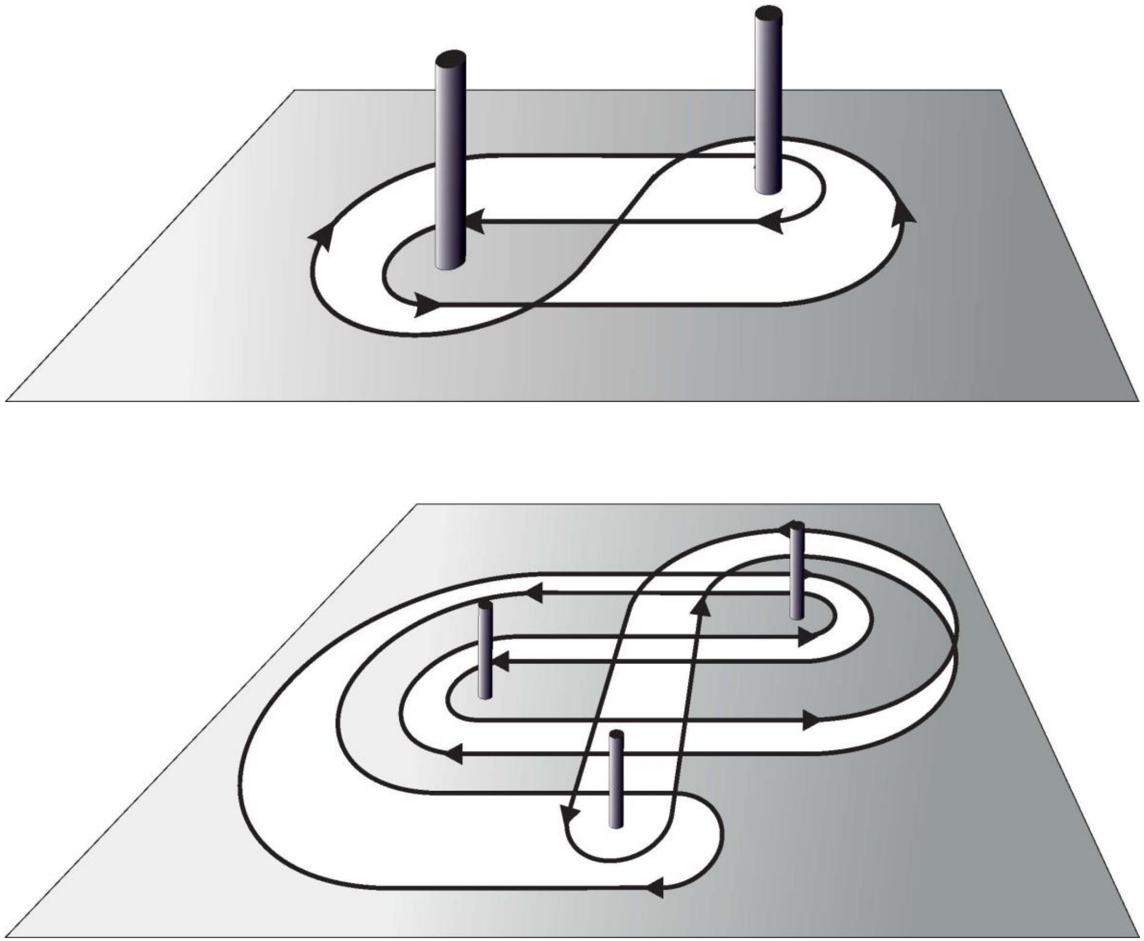


2D



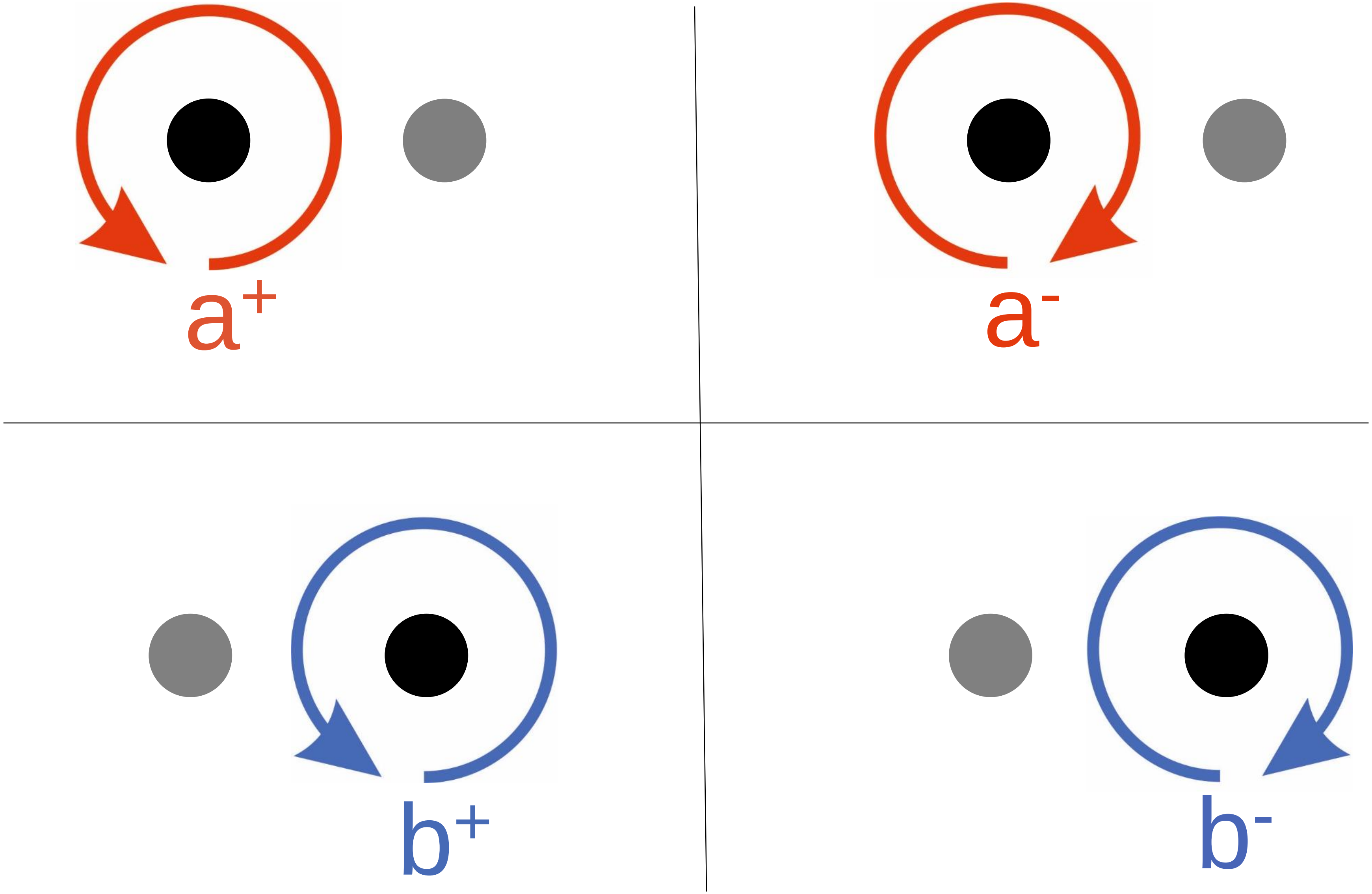
- Combinatorial approach: E. Helfand and D.S. Pearson (1983); M. Rubinstein and E. Helfand (1985)
- Random walks in a Non-Euclidean space: A. Khokhlov and S.N. (1985)
- Conformal approach for 2D entangled random walks: S.N. (1985-1988)

The key difficulty: non-commutativity

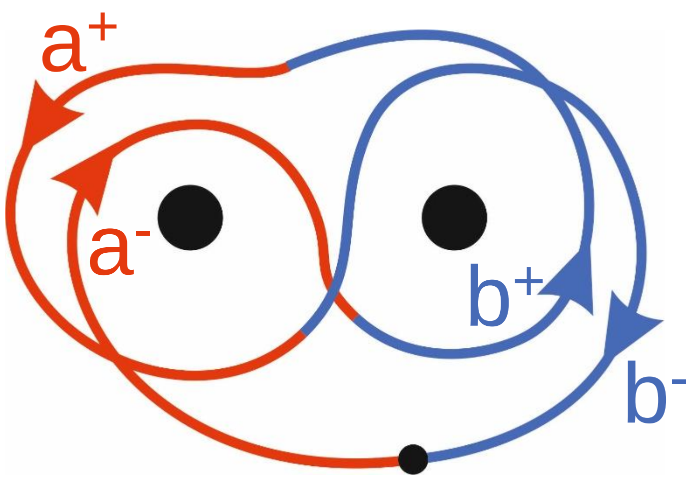


Commutative topological invariants, like the linking number, cannot distinguish these entangled states from the trivial one

Generators of turns

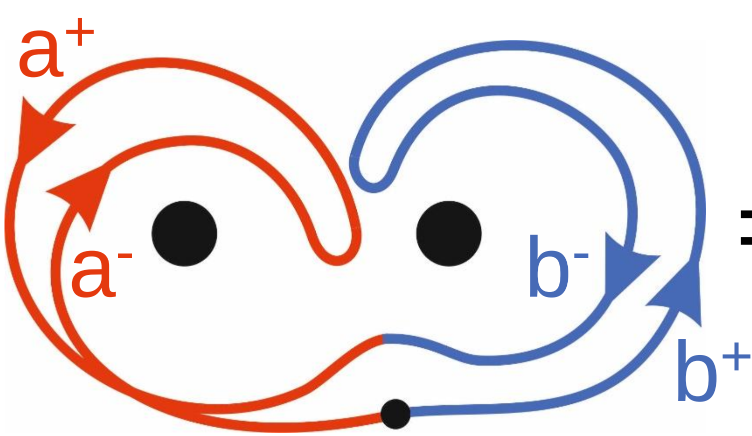


Non-commutative groups

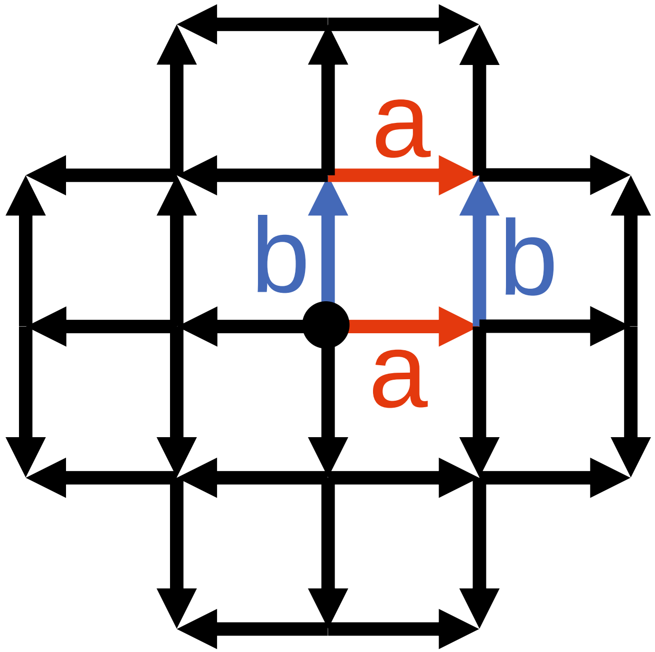


$$= a^- b^+ a^+ b^-$$

$$ab \neq ba$$



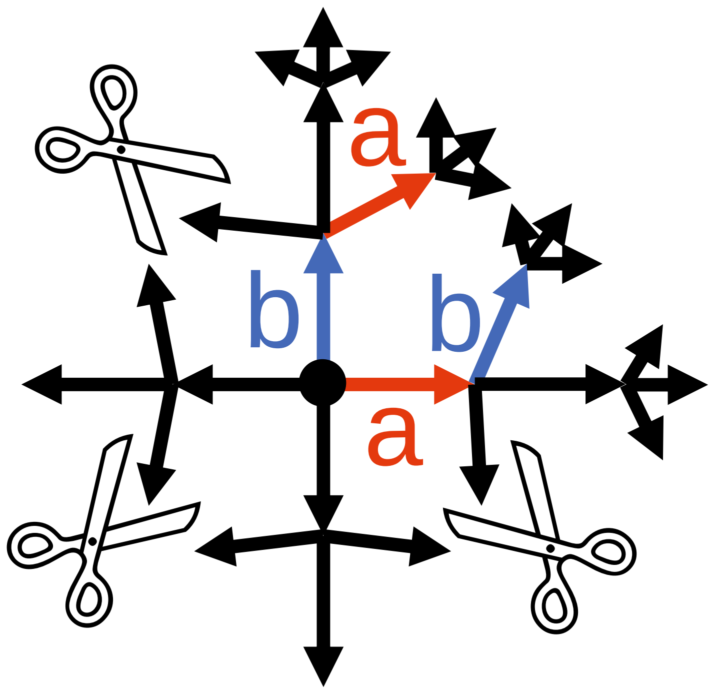
$$= \cancel{a^- a^+}\,\cancel{b^+ b^-} = 1$$



Cayley graph of a commutative group

$$a\,b = b\,a$$

Perimeter grows as ~ R



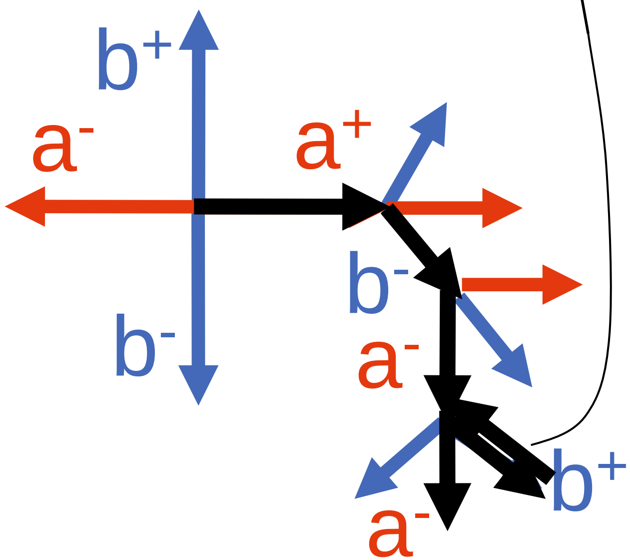
Cayley graph of a non-commutative (free) group

$$a\,b \neq b\,a$$

Perimeter grows as ~ 3^R

Metric structure of group of words

$$a^+b^-a^-\cancel{b^+b^-}a^- = a^+b^-a^-a^-$$



Distance from the root =
complexity of entanglement

Return to the origin =
full disentanglement

Probability to be unknotted, i.e. probability to return to the origin on a tree after N steps ~ $e^{-\mathrm{const}\,N}$

Probability for a length of a primitive path of an N-step random walk to be k is given by $P_N(k)$. The recursion for $P_N(k)$:

$$
\begin{aligned}
P_N(k) &= \tfrac{z-1}{z} P_{N-1}(k-1) + \tfrac{1}{z} P_{N-1}(k+1), \quad k \geq 2, \\
P_N(1) &= P_{N-1}(0) + \tfrac{1}{z} P_{N-1}(2), \\
P_N(0) &= \tfrac{1}{z} P_{N-1}(1), \\
P_0(k) &= \delta_{k,0}.
\end{aligned}
$$

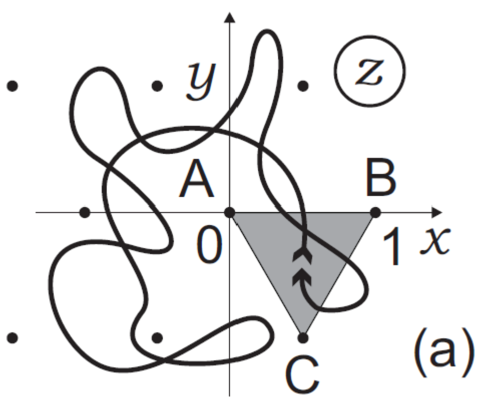
Continuous random walk in a lattice of obstacles

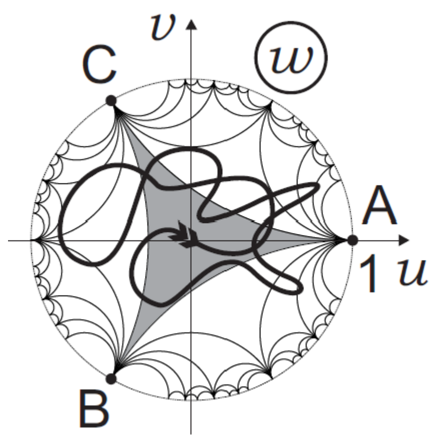
Find a probability that a continuous random walk of length t is not entangled with respect to the lattice of obstacles in the complex plane z.

Since the Laplace (Beltrami-Laplace) operator is conformally-invariant,

$$\Delta(w) = \partial_{uu}^2 + \partial_{vv}^2 = |w'(z)|^2(\partial_{xx}^2 + \partial_{yy}^2)$$

let us construct a conformal map of multi-punctured plane z to the "universal covering Riemann surface" w which is *free* of obstacles

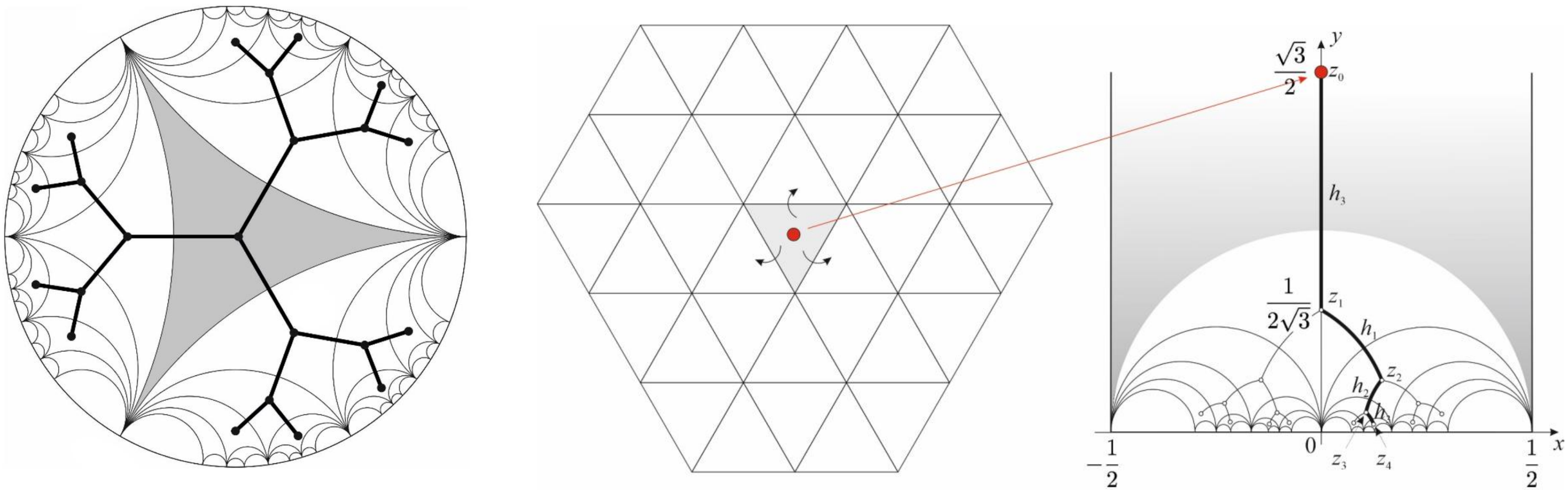




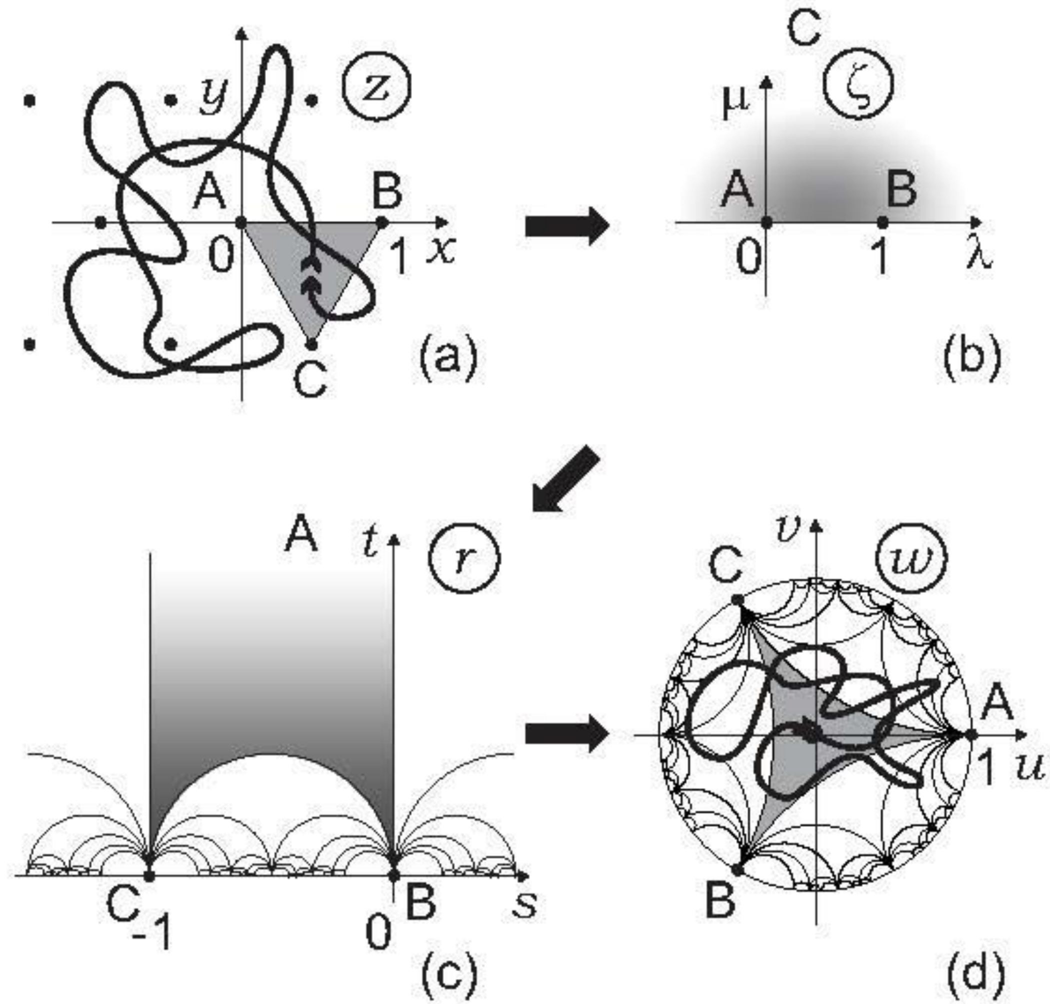
Diffusion equation under conformal transform can be written using Beltrami-Laplace operator

$$\partial_t P(w,t) - D|z'(w)|^{-2}(\partial_{uu}^2 + \partial_{vv}^2)P(w,t) = \delta(w - w_0)\delta(t)$$

Schematic connection between the tree and the Riemann surface free of obstacles obtained by the conformal transform



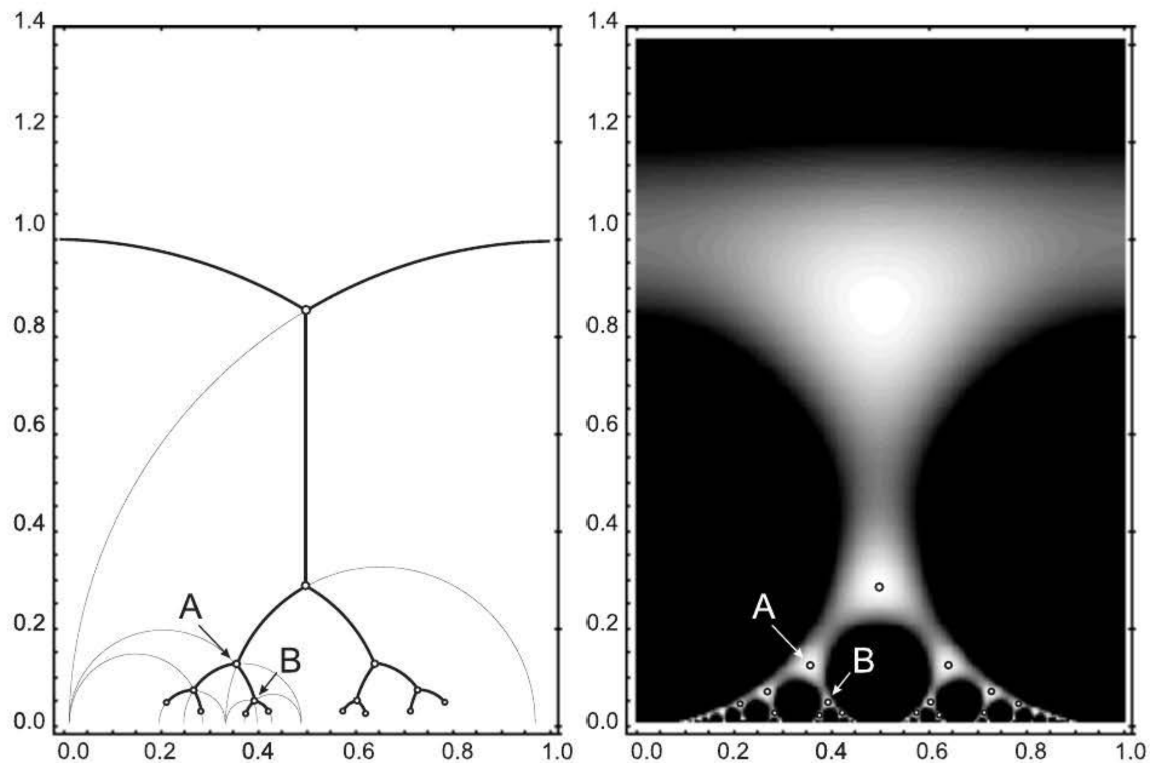
The map $z(w)$ is constructed in three sequential steps:



Define the function $f(z)$ as follows:

$$f(z) = C^{-1}\,|\eta(z)|\,(\mathrm{Im}\,z)^{1/4}$$

where $C = \left|\eta\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right|\,\left(\frac{\sqrt{3}}{2}\right)^{1/4} = 0.77230184....$ The function $f(z)$ has local maxima at all the points z_{cen} which are the centers of zero-agled triangles and only at these points.

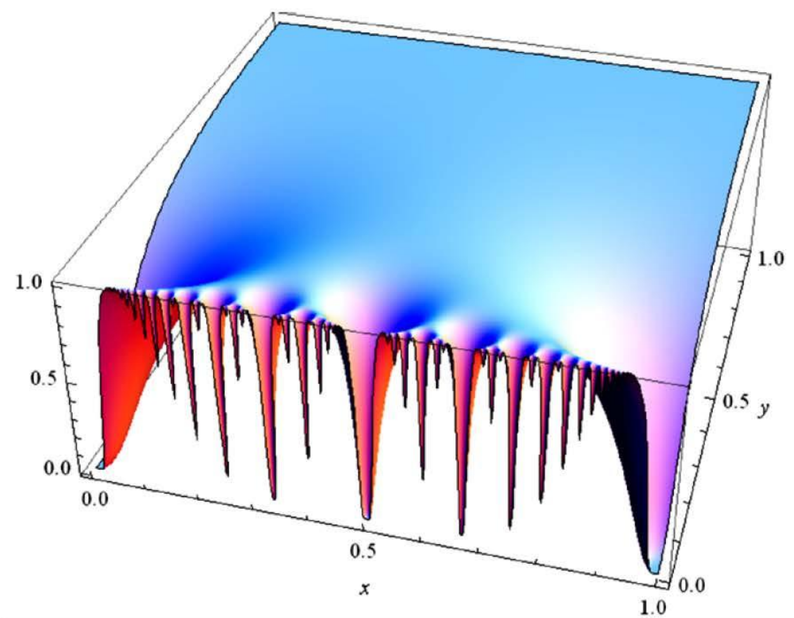


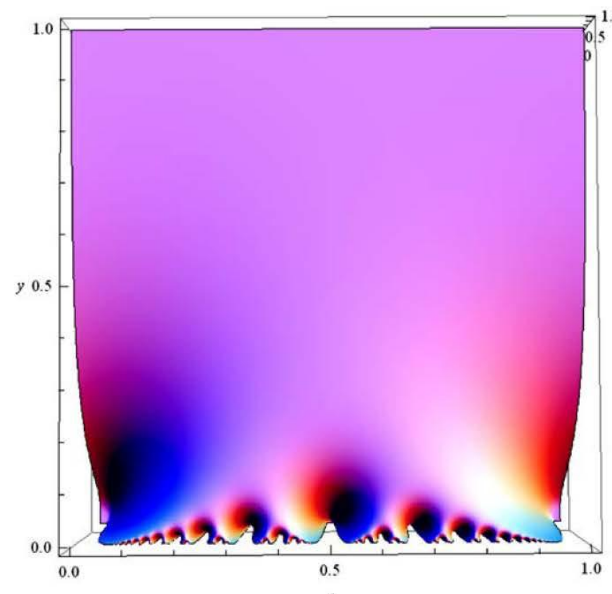
Reminder: *Dedekind η-function*

$$\eta(z) = e^{\pi i z/12} \prod_{n=0}^{\infty} (1 - e^{2\pi i n z}); \quad z = x + iy \quad (y > 0)$$

$$\eta(z+1) = e^{\pi i z/12}\, \eta(z) \qquad \eta\left(-\tfrac{1}{z}\right) = \sqrt{-i}\, \eta(z)$$

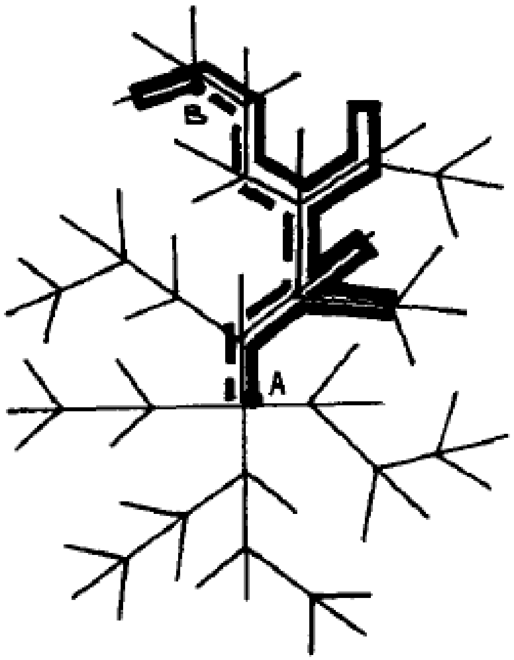
Define $h(z) = \text{const}\, |\, \eta(x+iy)\, |\, y^{1/4}$

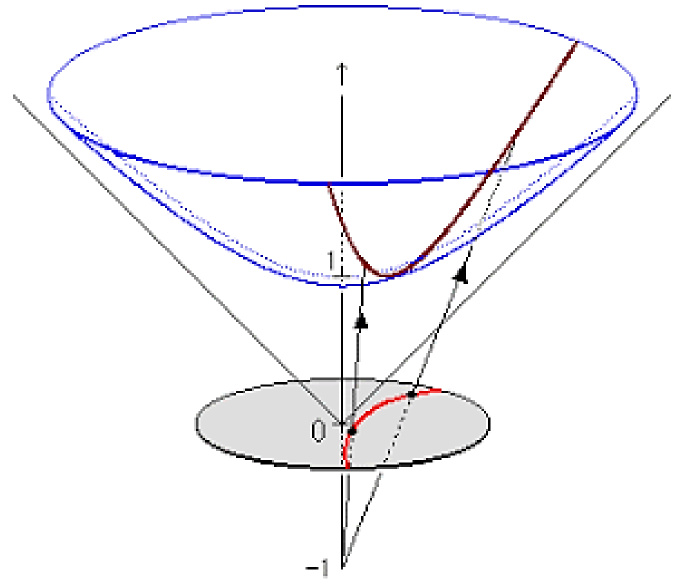




Isometric embedding of regular trees into the hyperbolic plane

M. Escher “Circle limit”





Hyperbolic metric in a Poincare disc and on a pseudosphere (stereographic projection of the unit disc)

$$ds^2 = \frac{d\varsigma d\varsigma^*}{(1-\varsigma\varsigma^*)^2} \; ; \qquad ds^2 = d\mu^2 + \sinh^2 \mu \, d\varphi^2$$

Random walks on hyperbolic surfaces

$$\frac{\partial}{\partial t} P(\mathbf{x},t) = D \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(\sqrt{g} \left(g^{-1} \right)_{ik} \frac{\partial}{\partial x_k} \right) P(\mathbf{x},t)$$

$$P(\mathbf{x}, t=0) = \delta(\mathbf{x}) \qquad \int \sqrt{g}\, P(\mathbf{x},t) d\mathbf{x} = 1$$

where the metric tensor is

$$\left\| g_{ik} \right\| = \left\| \begin{matrix} 1 & 0 \\ 0 & \sinh^2 \mu \end{matrix} \right\|; \quad g = \det \left\| g_{ik} \right\|;$$

$$ds^2 = d\mu^2 + \sinh^2 \mu \; d\varphi^2$$

Brownian bridges on hyperbolic surfaces

Probability to be in a specific point located at distance μ

$$P(\mu,t) = \frac{\exp(-tD/4)}{4\pi\sqrt{2\pi(tD)^3}} \int_{\mu}^{\infty} \frac{\xi \exp(-\xi^2/(4tD))}{\sqrt{\cosh\xi - \cosh\mu}} d\xi$$

$$\sim \frac{\exp(-tD/4)}{4\pi tD} \left(\frac{\mu}{\sinh\mu} \right)^{1/2} \exp\left(-\frac{\mu^2}{4tD} \right)$$

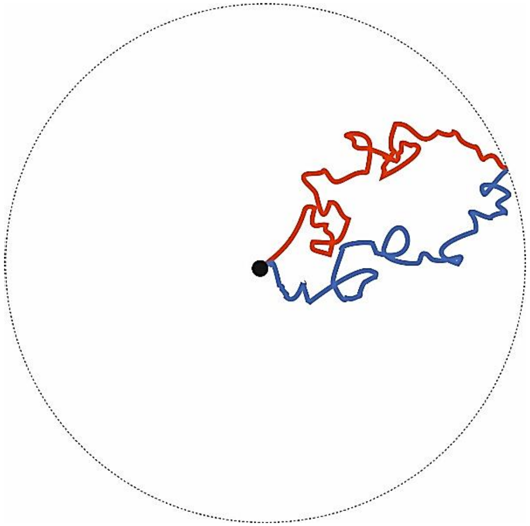
Probability to be in any point located at μ

$$Q(\mu,t) = 2\pi P(\mu,t) \sinh\mu$$

Conditional probability for a Brownian bridge to be in any point located at distance μ

$$Q(\mu,t\,|\,0,N)=\frac{P(\mu,t)P(\mu,N-t)\sinh\mu}{P(0,t)}$$

$$\sim\frac{N}{4\pi Dt(N-t)}\mu\exp\left(-\frac{\mu^2}{4D}\left(\frac{1}{t}+\frac{1}{N-t}\right)\right)$$



- A. Khokhlov, S. N. (1985)
- S.N., A. Semenov, M. Koleva (1987)
- S.N., Y. Sinai (1991)
- P. Bougerol, T. Jeulin (1999)

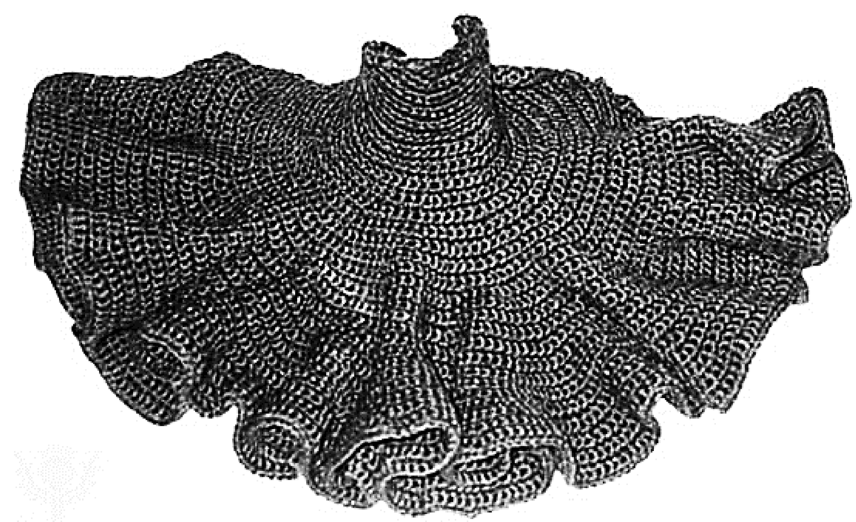
hyperbolic disc or preudosphere

The curvature is effectively “killed” and the Brownian bridge statistics in hyperbolic geometry is as in a flat space

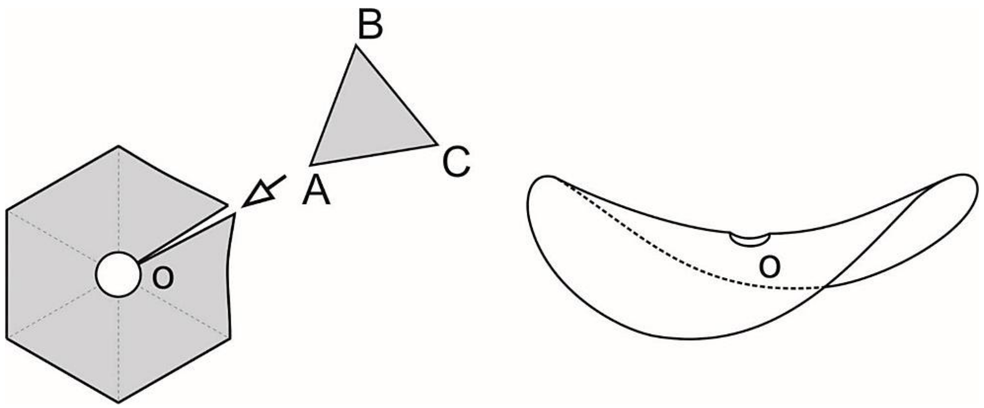
This fact is crucial for conditional statistics of knots

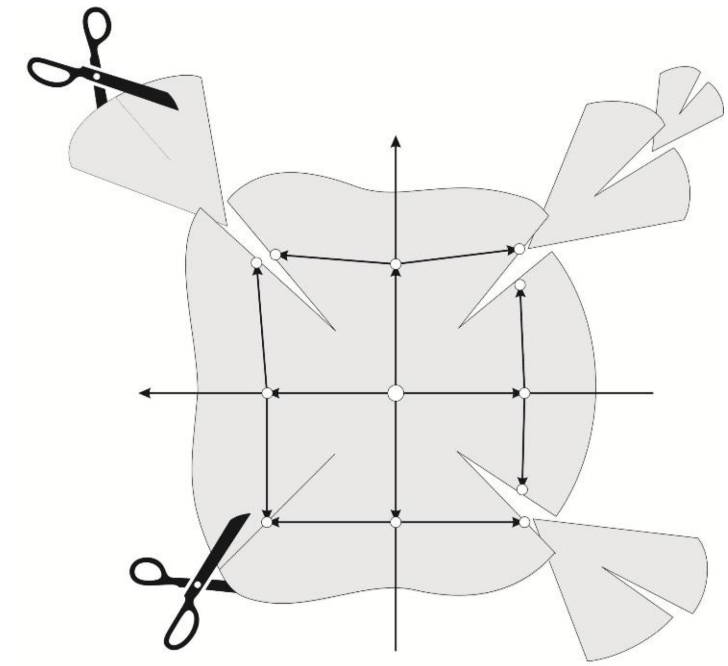
2. Isometric embedding of an uniform unbound Cayley tree in 3D and a profile of a salad leaf

Isometric embedding of hyperbolically growing tissue in a 3D space



How to describe the profile?

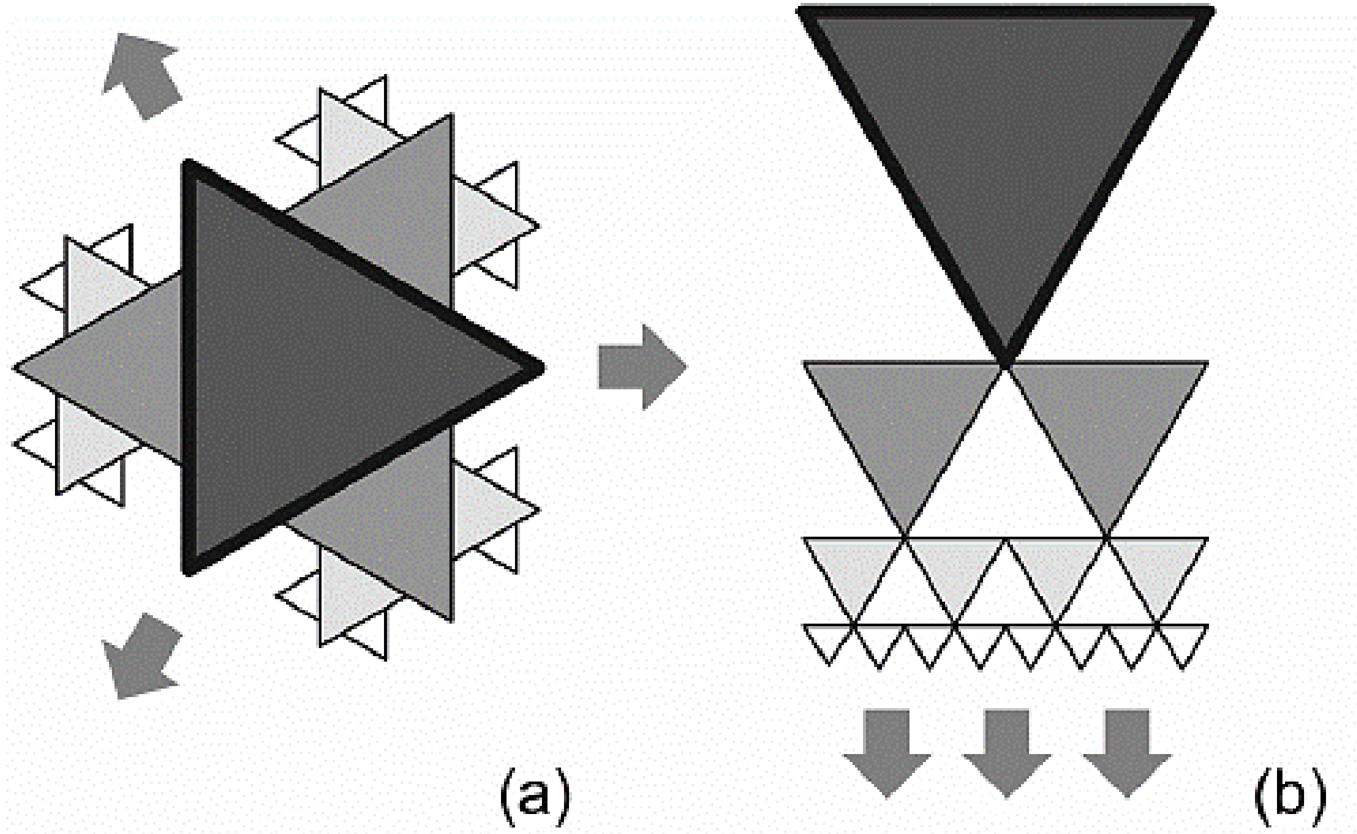




Jupe à godets

R. Voituriez, S.N., J. Phys. A, 2001

Exponential proliferation of cells in a thin slit



Growth in a disc

Growth in a strip

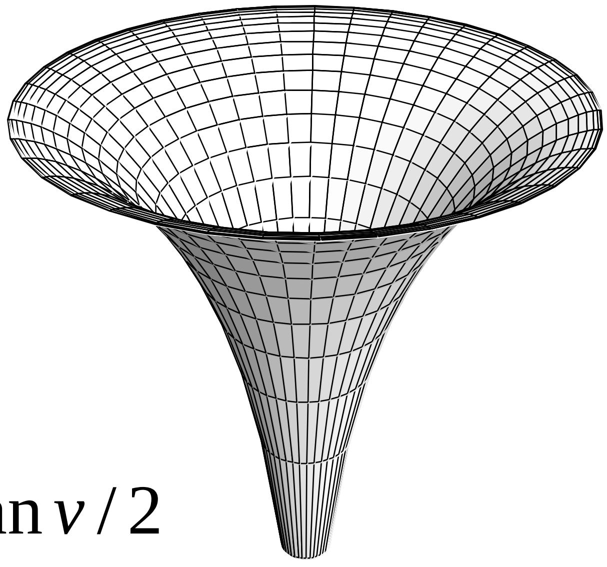
When we open the slit, the material relaxes into the 3D structure

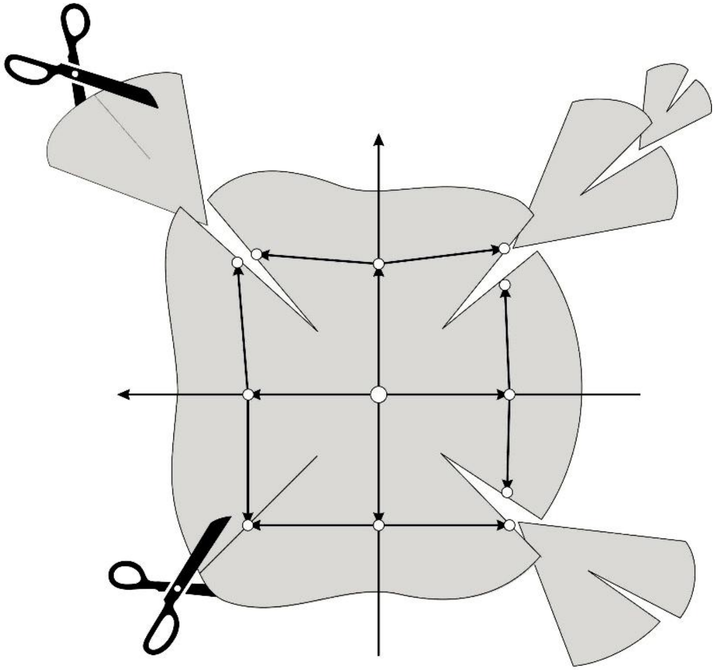
Conjecture: buckling occurs as a conflict between internal geometry of object and geometry of space of embedding

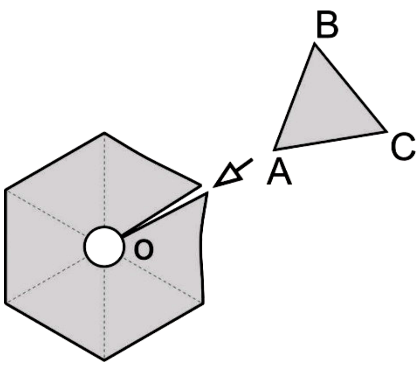
Hyperbolic surfaces with **a boundary** can be isometrically embedded in a Euclidean space.

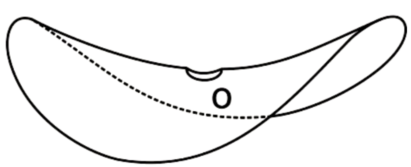
The revolution of the *tractrix* is an example of such an embedding:

$$\begin{cases} x = \cos u \sin u \\ y = \sin u \sin v \\ z = \cos v + \log \tan v / 2 \end{cases}$$









$$ds^2 = \frac{d\varsigma\, d\varsigma^*}{(1-\varsigma\varsigma^*)^2}$$

Lobachevsky plane

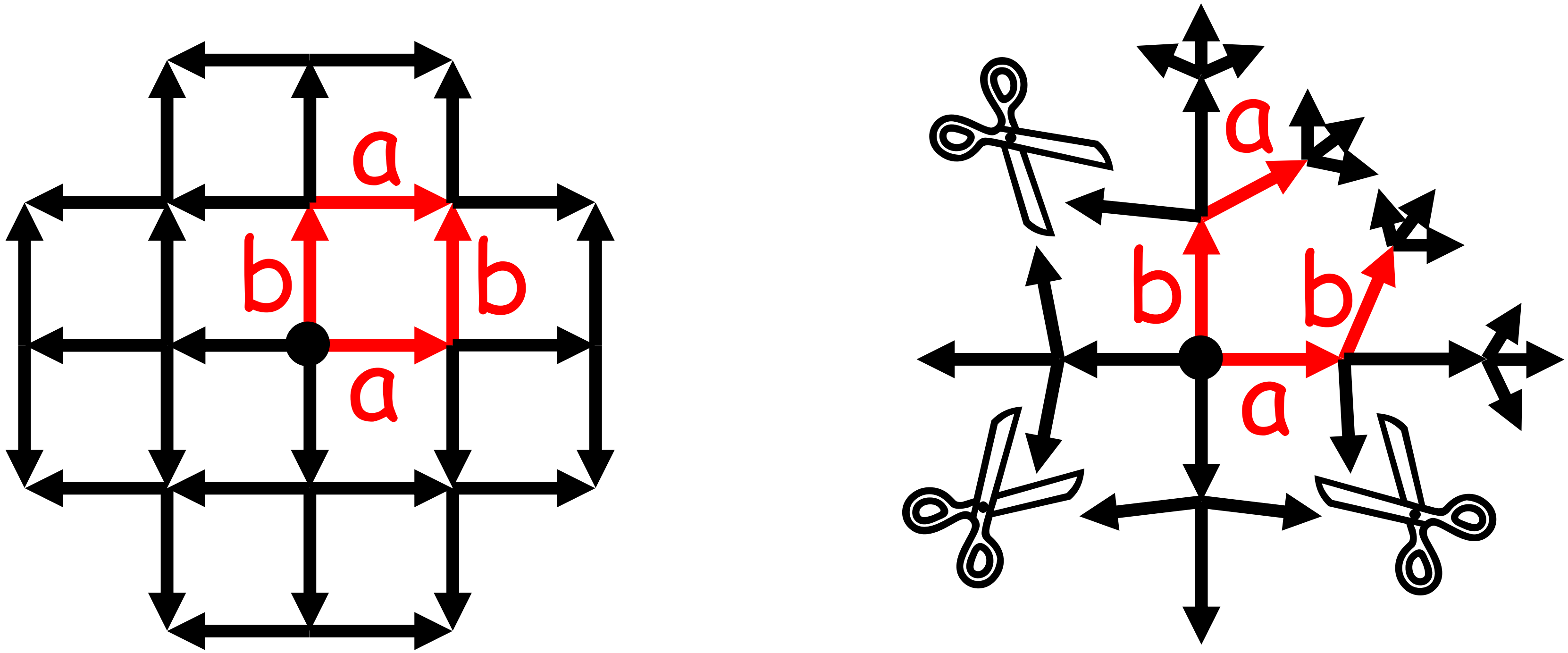
Formulation of the problem

- Exponentially growing colony (hyperbolic structure) admits Cayley trees as possible discretizations.

- The Cayley trees cover the hyperbolic surface *isometrically*, i.e. without gaps and selfintersections, preserving angles and distances.

- Our goal is an embedding a Cayley tree into a 3D Euclidean space with a signature {+1,+1,+1}.

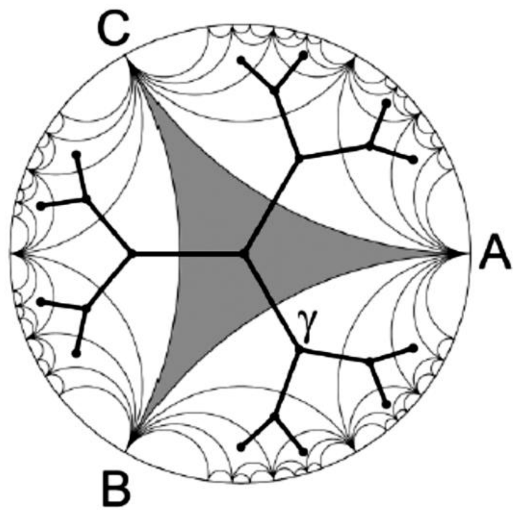
- Hilbert theorem prohibits embedding of **open** hyperbolic surface into Euclidean space smoothly.

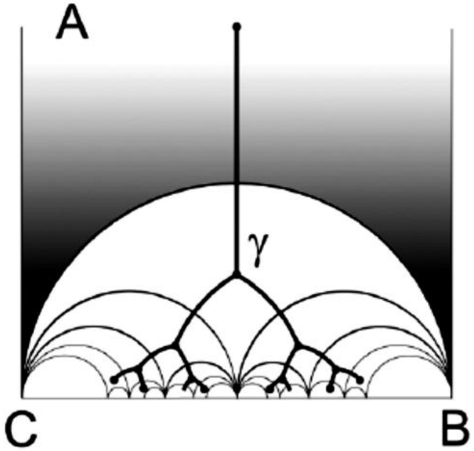


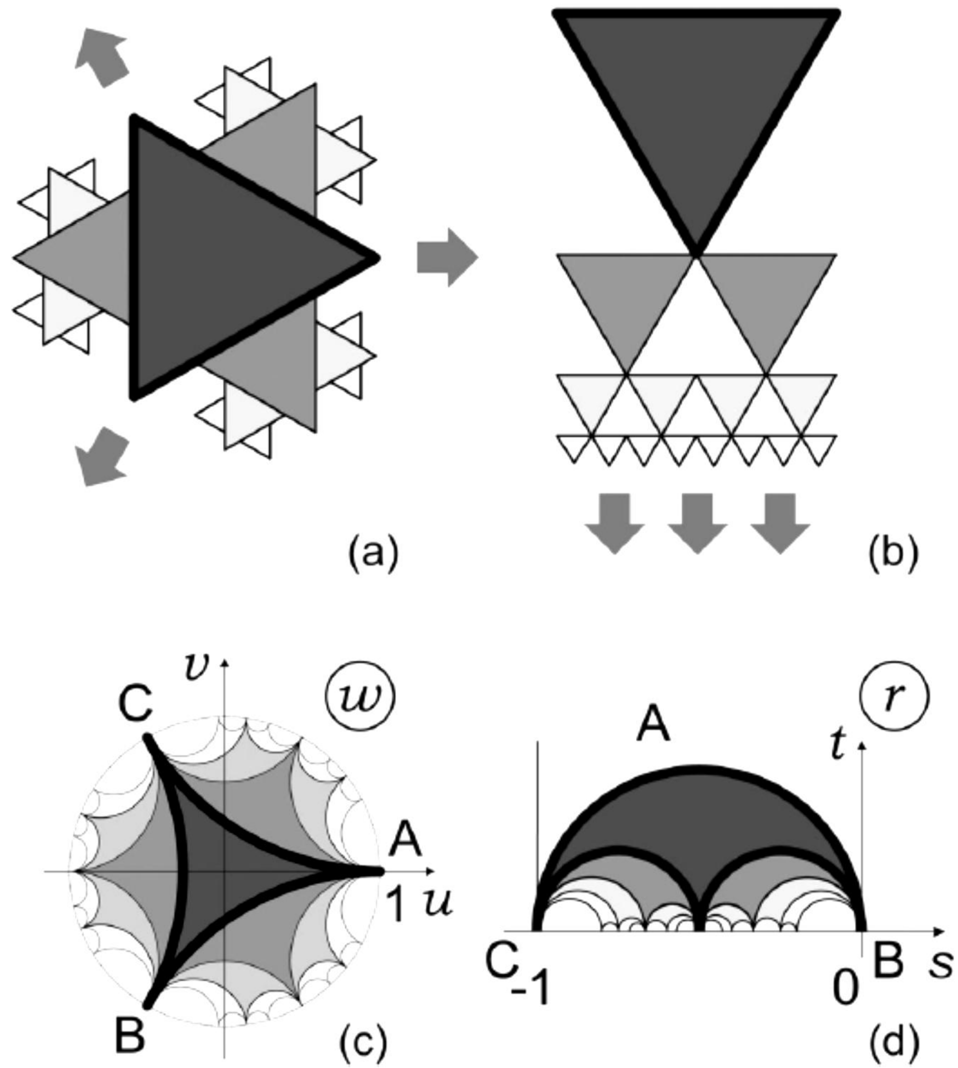
The relief of the surface is encoded in the *coefficient of deformation*, coinciding with the Jacobian $J(\zeta)$ of the conformal transform $z(\zeta)$, where

$$J(\varsigma) = \left| dz / d\varsigma \right|^2$$

Isometric embedding of a Cayley tree into Poincare disc and a strip

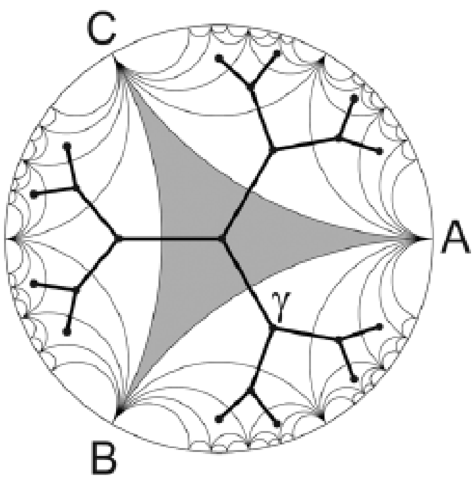


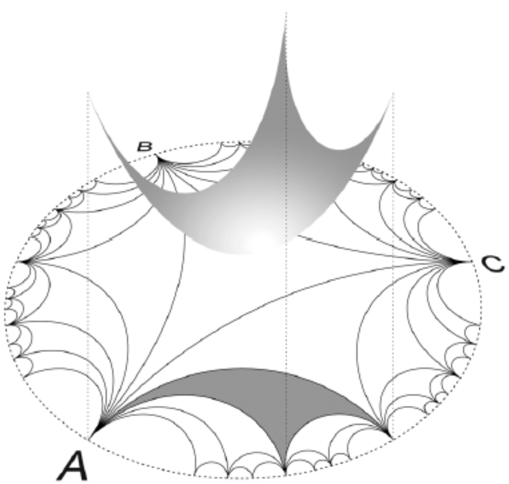




Optimal profile – is the surface in which we can *isometrically* embed exponentially growing graph

S. N., K. Polovnikov, Soft Matter, 2017





The metric *ds²* of a 2D surface parametrized by *(u,v)*, is given by the coefficients

$$E = \mathbf{r}_u{}^2, \quad F = \mathbf{r}_v{}^2, \quad G = (\mathbf{r}_u, \mathbf{r}_v)$$

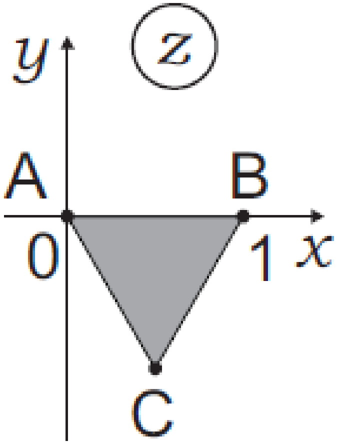
of the first quadratic form of this surface

$$\mathrm{d}s^2 = E\mathrm{d}u^2 + 2F\,\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$$

The surface area then reads

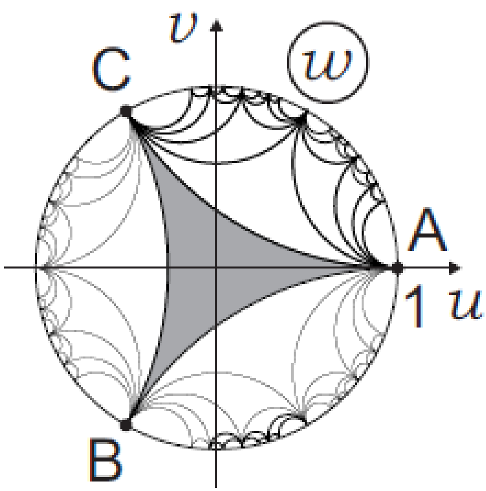
$$\mathrm{d}S = \sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v$$

Surface embedded in 3D has the same area as in the Poincare disc



$$S_{\mathrm{ABC}} = \int_{\Delta\mathrm{ABC}} \mathrm{d}x\mathrm{d}y = \mathrm{const}$$

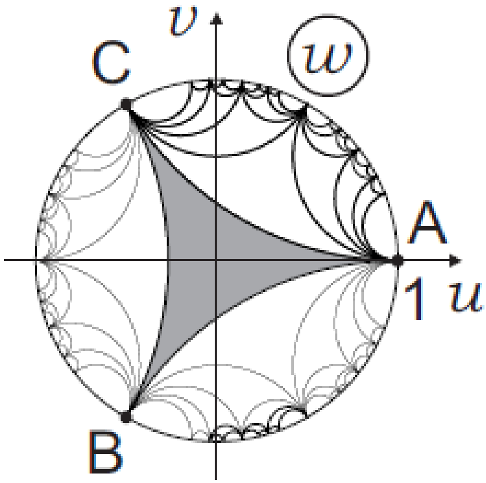
$$S_{\mathrm{ABC}} = \int_{\Delta\mathrm{ABC}} |J(z,w)|\mathrm{d}u\mathrm{d}v; \quad J(z,w) = \begin{vmatrix} \partial_u x & \partial_u y \\ \partial_v x & \partial_v y \end{vmatrix}$$



If $z(w)$ is holomorphic, the Cauchy-Riemann conditions provide

$$J(w) = \left|\frac{\mathrm{d}z(w)}{\mathrm{d}w}\right|^2 \equiv |z'(w)|^2$$

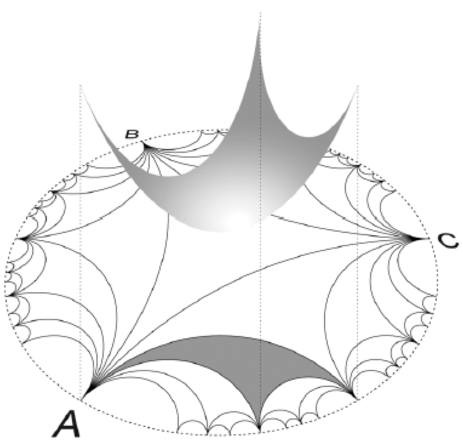
Surface embedded in 3D has the same area as in the Poincare disc



$$S_{A'} = \int_{\triangle A'} |J(z,w)| du dv$$

Imposing the condition for a surface to be a *function* above *(u,v)*, then we can write the surface element in curvilinear coordinates

$$J(u,v) = \sqrt{1 + (\partial_u f)^2 + (\partial_v f)^2}$$



In polar coordinates of the complex w-domain,

$$\{(\rho,\phi): u = \rho\cos\phi,\ v = \rho\sin\phi\}$$

we arrive at the nonlinear partial differential equation for the surface profile $f(\rho,\phi)$ above w:

$$\left(\partial_\rho f(\rho,\phi)\right)^2 + \frac{1}{\rho^2}\left(\partial_\phi f(\rho,\phi)\right)^2 = |z'(w)|^4 - 1$$

In the case of the strip domain we arrive at the equation

$$(\partial_s f(s,t))^2 + (\partial_t f(s,t))^2 = |z'(r)|^4 - 1$$

In general, the relief of the surface $f\,(u,v)$ is defined by the *eikonal* equation for a wavefront, describing the light propagation according to the Huygens principle in the unit disk:

$$(\nabla S(\mathbf{x}))^2 = n^2(\mathbf{x})$$

with the refraction coefficient

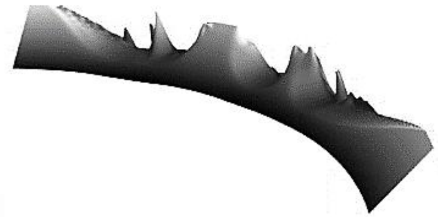
$$n(w) = \sqrt{|z'(w)|^4 - 1}$$

In our case we have to solve the equation for the function $f(u,v)$ given by the Jacobian $|J(r)|^2$ of the conformal transform

$$J(r) = |z'(r)|^2 = \frac{h^2}{a^2}|\eta(r)|^8$$

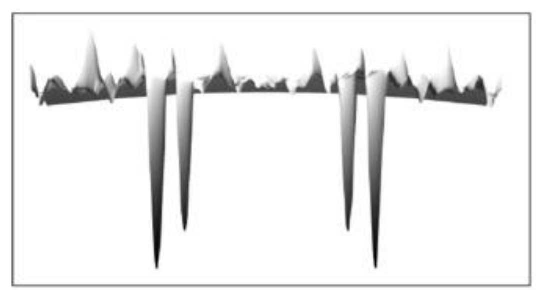
Rigid and flexible circular surfaces

The rigidity in our geometric approach is controlled by the parameter a – the size of elementary flat domain



1) If $a \geq 1$, the surface is rigid

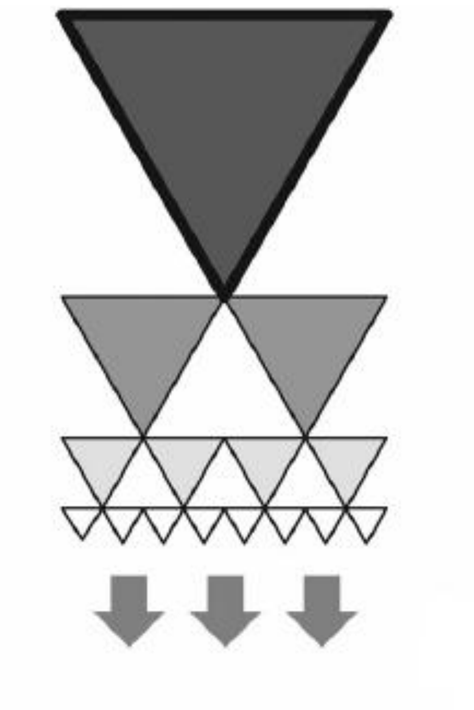
2) If $0 < a << 1$, the surface is flexible

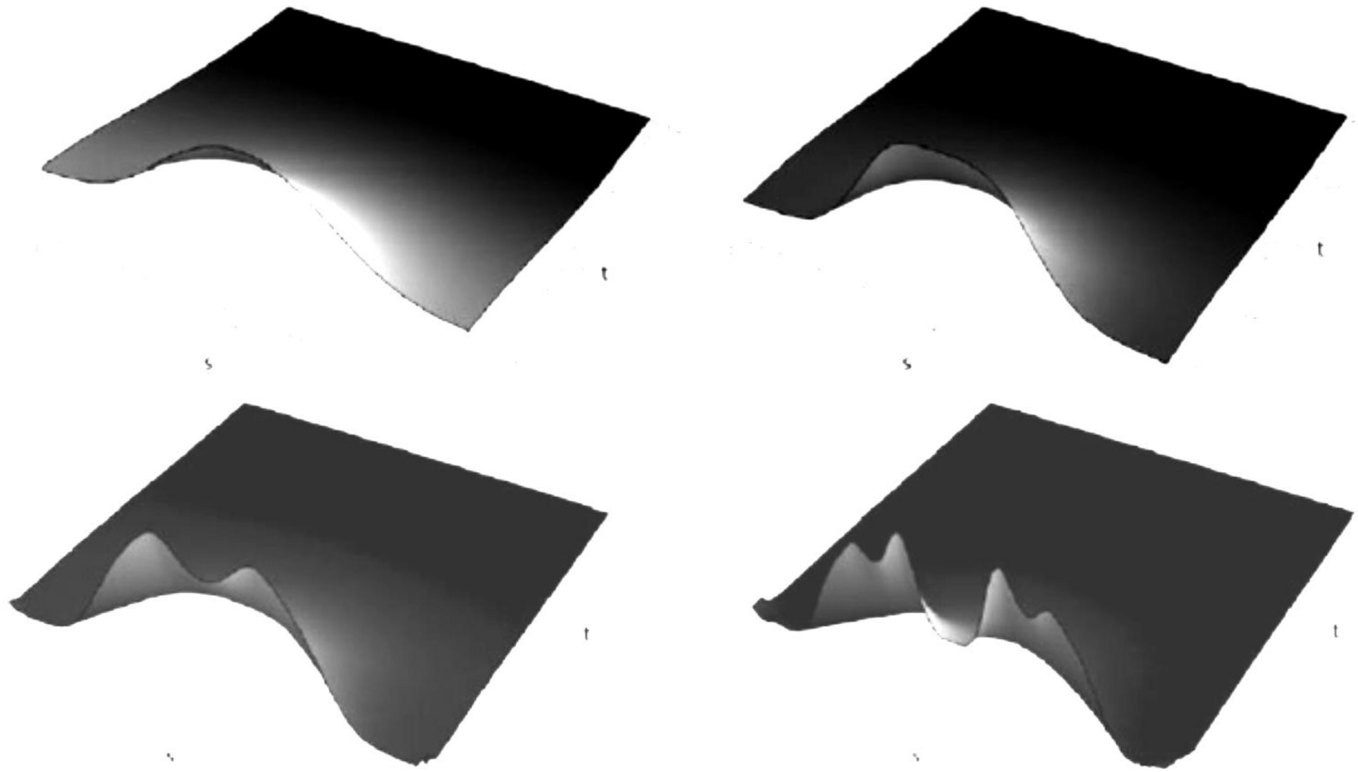


a=0.07

a=0.14

Buckling in a strip



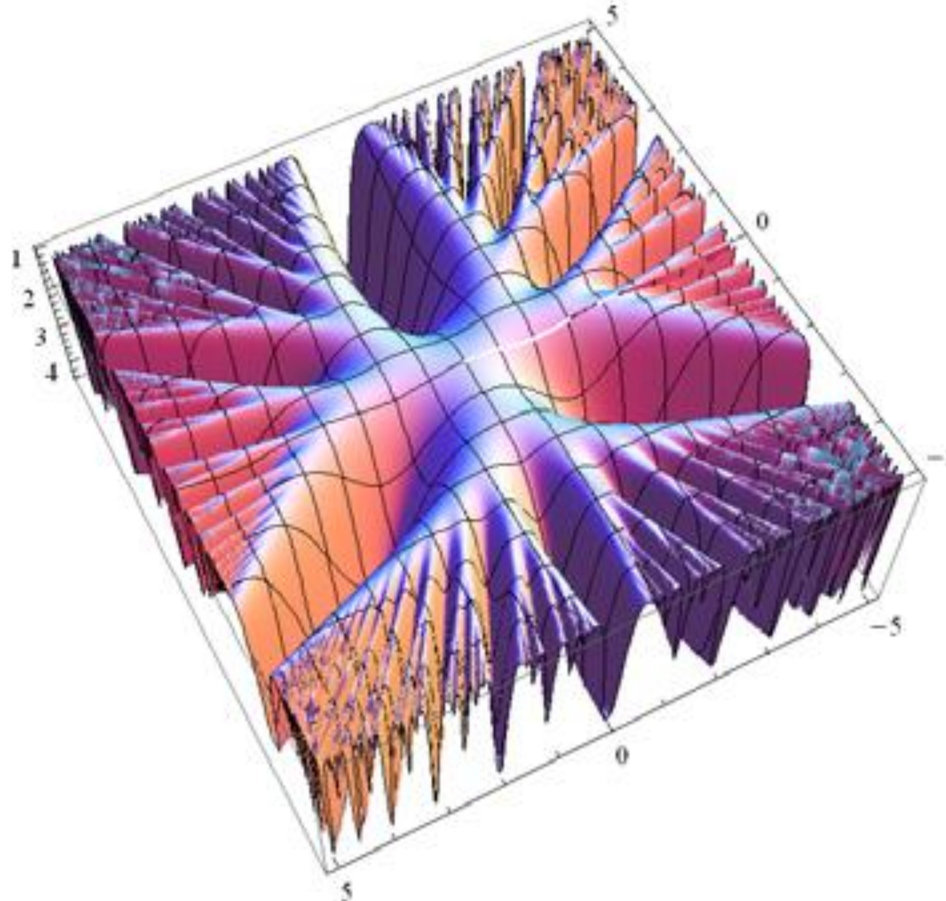


Formation of hierarchical folds due to ultrametricity of Dedekind η-function

Other manifestations of hyperbolic geometry in living nature

Coral

Relief of a normalized modular function



Literature

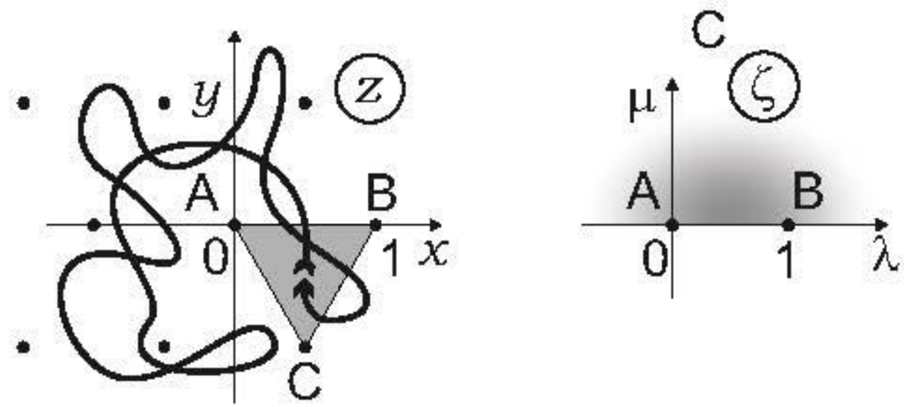
On random walks in lattices of obstacles:

1. A.R. Khokhlov, S.K. Nechaev, Polymer chain in an array of obstacles, Phys. Lett. (A), 112 (1985), 156-160.
2. S.K. Nechaev, Topological properties of a 2D-polymer chain in the lattice of obstacles, J. Phys. (A): Math. Gen., 21 (1988), 3659-3671
3. S.K. Nechaev, Concepts of polymer statistical topology, arXiv:1608.06529, in “Topology in Condensed Matter Physics” (ed. S. M. Bhattacharjee) (2017) – review with all relevant references

On the shape of salad leafs:

1. S. Nechaev, R. Voituriez, On the plants leaves boundary, "jupe a godets" and conformal embeddings, J.Phys. (A): Math. Gen. 34 (2001) 11069-11082
2. S. Nechaev, K. Polovnikov, From geometric optics to plants: eikonal equation for buckling, Soft Matter, 13 (2017), 1420-1429

First step: mapping to upper half-plane



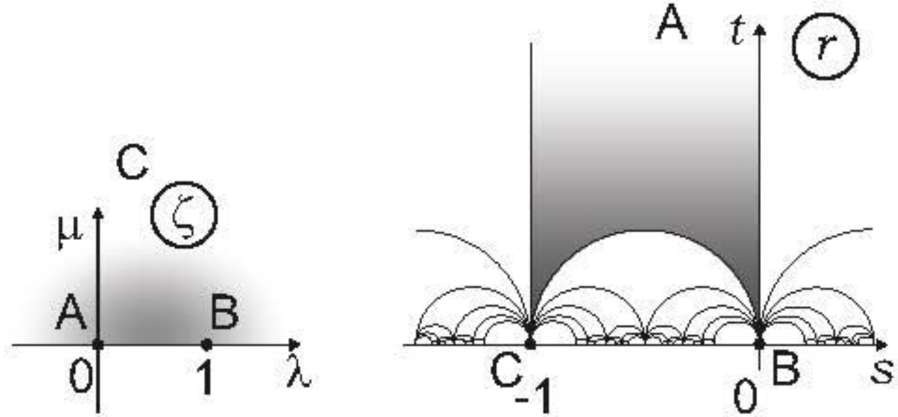
Map ABC to half-plane ζ via Schwarz-Christoffel integral:

$$z(\zeta) = \frac{\Gamma(\frac{2}{3})}{\Gamma^2(\frac{1}{3})} \int_0^{\zeta} \frac{d\xi}{\xi^{2/3}(1-\xi)^{2/3}}$$

Points correspondence:

- Vertex A ($z = 0$) $\to \zeta = 0$
- Vertex B ($z = 1$) $\to \zeta = 1$
- Vertex C ($z = e^{-i\pi/3}$) $\to \zeta = \infty$

Second Step: mapping to fundamental domain



Map upper half-plane ζ to circular triangle ABC in r-plane. We use ratio of solutions to Picard-Fuchs equation:

$$r(\zeta) = \frac{\phi_1(\zeta)}{\phi_2(\zeta)}$$

The differential equation (hypergeometric type):

$$\zeta(\zeta - 1)\phi'' + \big((a + b + 1)\zeta - c\big)\phi' + ab\phi = 0$$

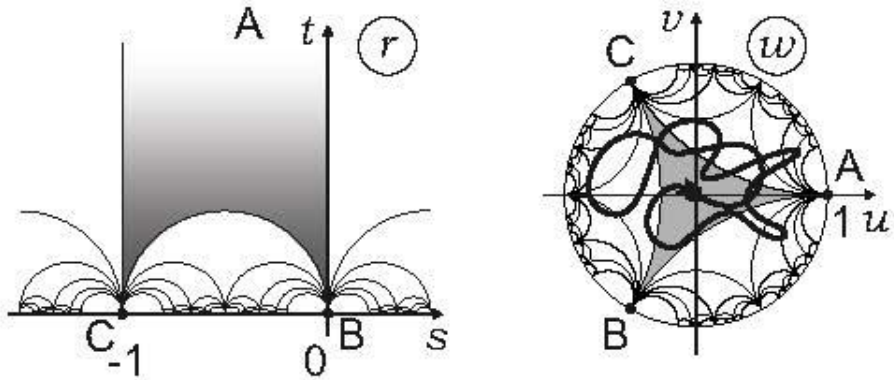
Function $r(\zeta)$ conformally maps the hyperbolic triangle ABC, lying in the upper halfplane of w onto the upper halfplane of ζ. The sides of the triangle ABC are arcs of semicircles orthogonal to the real axis, and the angles are $\{\alpha_0 = \pi|c-1|, \alpha_1 = \pi|a+b-c|, \alpha_\infty = \pi|a-b|\}$. Choosing $\alpha_\infty = 0$ and $\alpha_0 = \alpha_1 = \alpha$, we can express the parameters (a, b, c) in terms of α: $\{\alpha_0, \alpha_1, \alpha_\infty\} = \{\alpha, \alpha, 0\}$ with $a = b = \frac{\alpha}{\pi} + \frac{1}{2}, c = \frac{\alpha}{\pi} + 1$.

$$\zeta(\zeta-1)\phi''(\zeta) + \left(\frac{\alpha}{\pi}+1\right)(2\zeta-1)\phi'(\zeta) + \left(\frac{\alpha}{\pi}+\frac{1}{2}\right)^2\phi(\zeta) = 0$$

where $\alpha = \frac{\pi}{m}$ and $m = 3, 4, ...\infty$. For $\alpha = 0$ equation takes a simple form, known as Legendre hypergeometric equation. A pair of fundamental solutions of Legendre equation are

$$\phi_1 = F(\tfrac{1}{2}, \tfrac{1}{2}, 1, \zeta), \quad \phi_2 = iF(\tfrac{1}{2}, \tfrac{1}{2}, 1, 1-\zeta)$$

Final mapping to unit disc



Map the r-plane to unit disc w via Möbius transformation:

$$r(w) = e^{-i\pi/3}\frac{e^{2i\pi/3} - w}{1 - w} - 1$$

The composite mapping's Jacobian involves:

$$J(z(w)) = 3h^2 \frac{|\eta(r(w))|^8}{|1 - w|^4}$$

Dedekind η function appears naturally:

$$\eta(w) = e^{\pi i w/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n w})$$