The 15th School of Mesoscopic Physics:
Fundamentals of Quantum Science & Technology

Two routes to understand mesoscopic superconductivity

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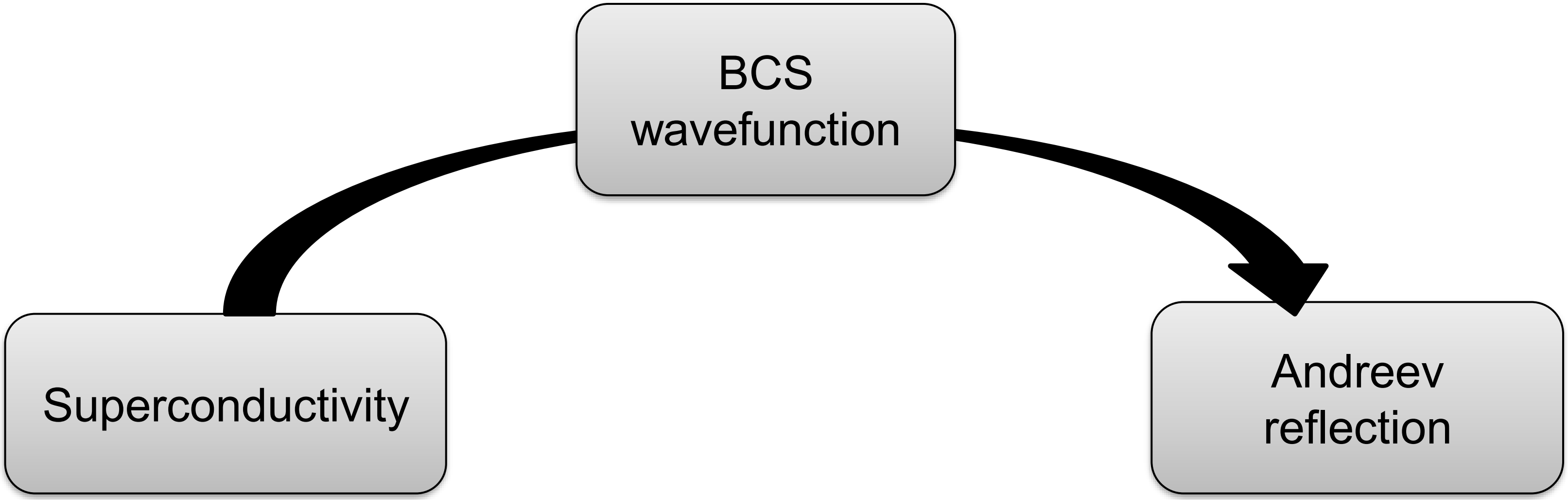
In this lecture

- **Route I:** superconductivity in historical order
  → What is superconductivity?
  → Theoretical description of superconductivity
  → Josephson junction & its descriptions

- **Route II:** mesoscopic superconductors from AR
  → SC as a black box providing Andreev reflections (ARs)
  → Bogoliubov de Genne (BdG) Hamiltonian
  → Ground state of a metal with AR = BCS wavefunction
  → Josephson junction & its theoretical descriptions

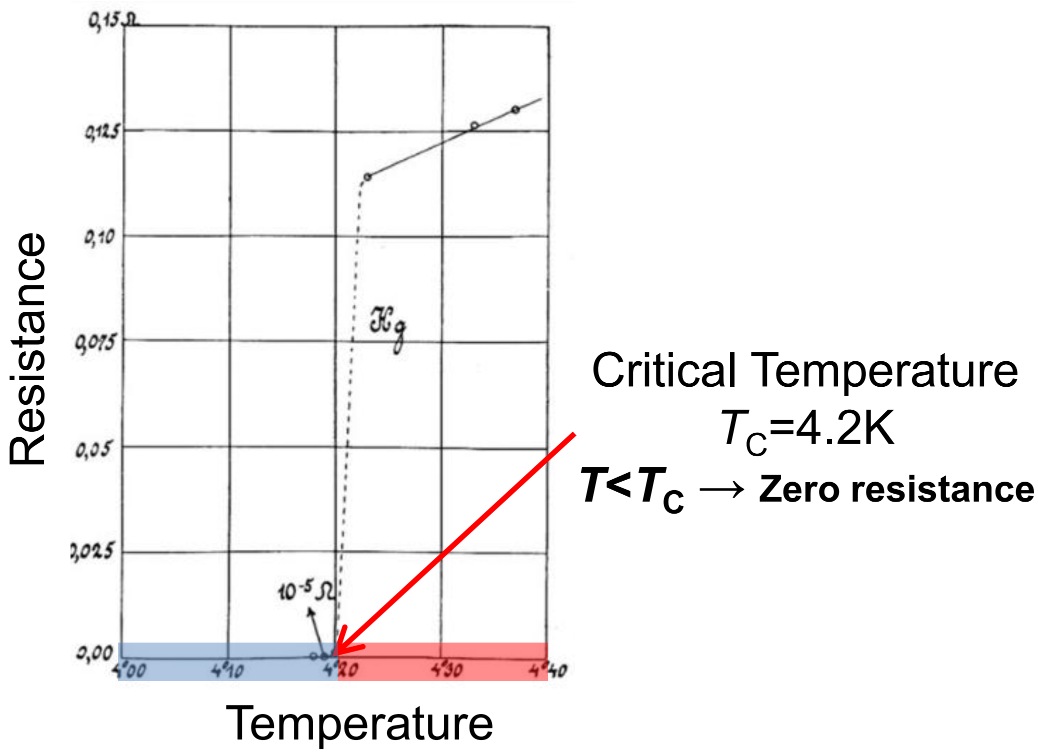
- **Beyond the conventional *s*-wave superconductivity**
  → What there are under the carpet

Route I:
superconductivity in historical order



Route I: superconductivity in historical order

- **Phenomenology**
  → Vanishing Resistance



H. K. Onnes, Commun. Phys. Lab.**12**,120, (1911)

Route I:
superconductivity in historical order

- **Phenomenology**
  → Vanishing Resistance

Existed understanding of resistance at low temp. at that time: Drude model

$$\rho = \frac{m}{ne^2\tau}$$

$\bar{v}_e\tau = l$ & $l = 1/n_{\text{ion}}\sigma$ but

$\bar{v}_e \propto \sqrt{T}$ & $\sigma \propto \langle u^2 \rangle \propto T$, i.e.,

$\rho \propto \frac{1}{\tau} \propto \frac{\bar{v}_e}{l} \propto \bar{v}_e\sigma$. $\therefore \rho \propto T^{\frac{3}{2}}$

Modern understanding of resistance of normal metals

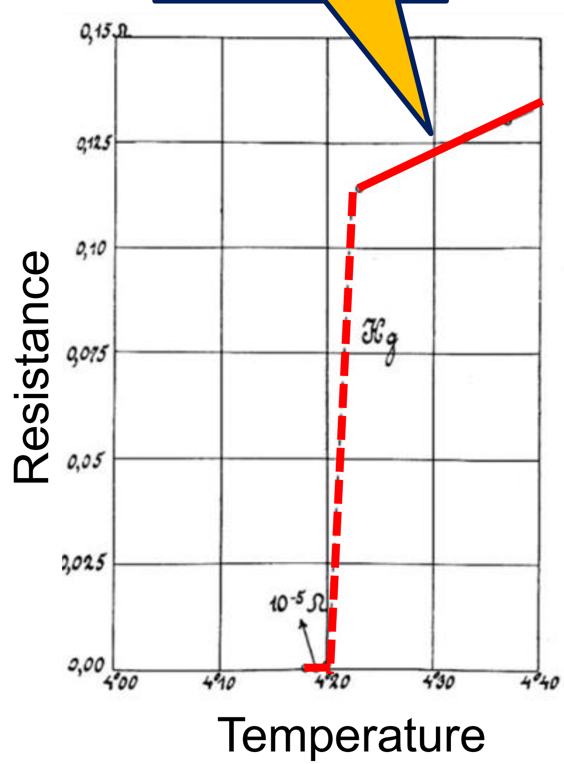
$$\rho = \rho_0 + \rho_{\text{ee}} + \rho_{\text{ep}} + \cdots$$

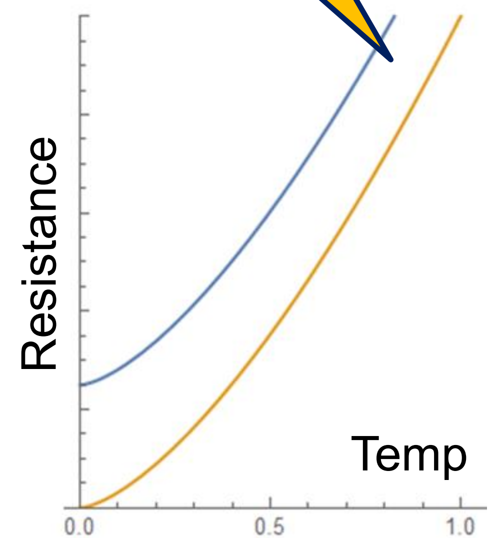
$\rho \approx \rho_0 + cT$ **(high T)**

$\rho \approx \rho_0 + aT^2 + bT^5$ **(low T)**

This is the case!?

Which is the case?



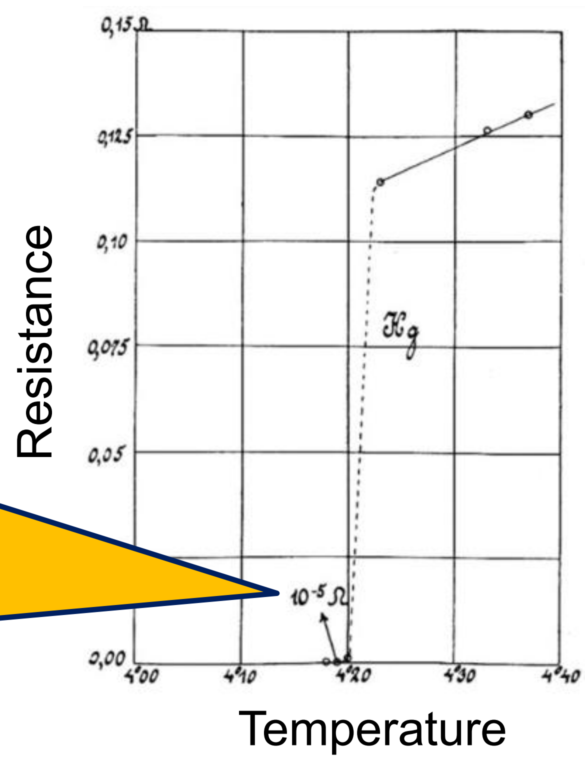


H. K. Onnes, Commun. Phys. Lab.**12**,120, (1911)

Route I: superconductivity in historical order

- **Phenomenology**
  → Vanishing Resistance

Is a superconductor merely an extremely good conductor
<u>(10^{-5} Ω was observed),</u>
or is it a signature of a thermodynamic phase transition?



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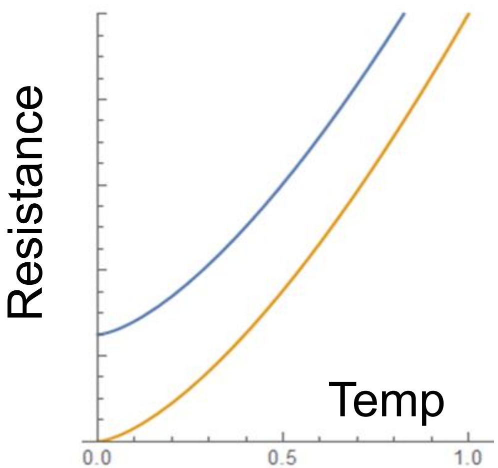
Route I: superconductivity in historical order

- **Phenomenology**
  → Is a perfect conductor all about the superconductivity?

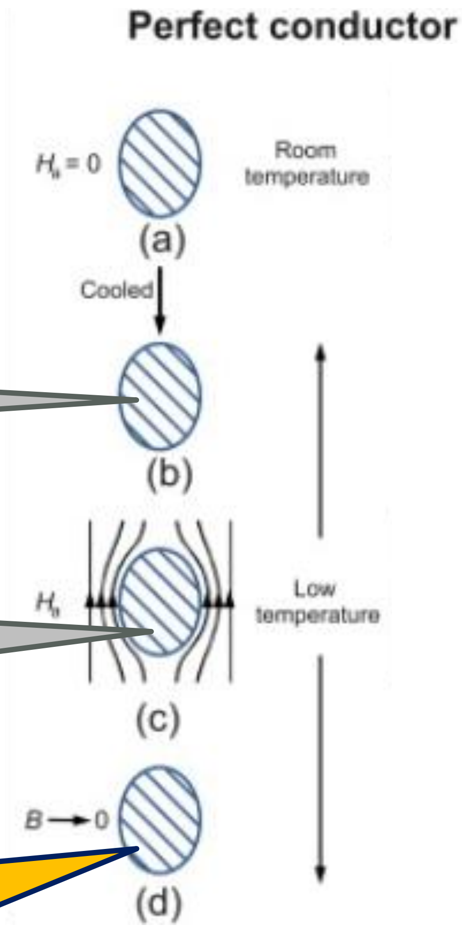


Premature $\rho \propto T^{\frac{3}{2}}$

Cool down first, $\rho \approx 0$

Apply B-field later: magnetic flux is fixed

Final state @ Low temp. after turning off B-field



Route I: superconductivity in historical order

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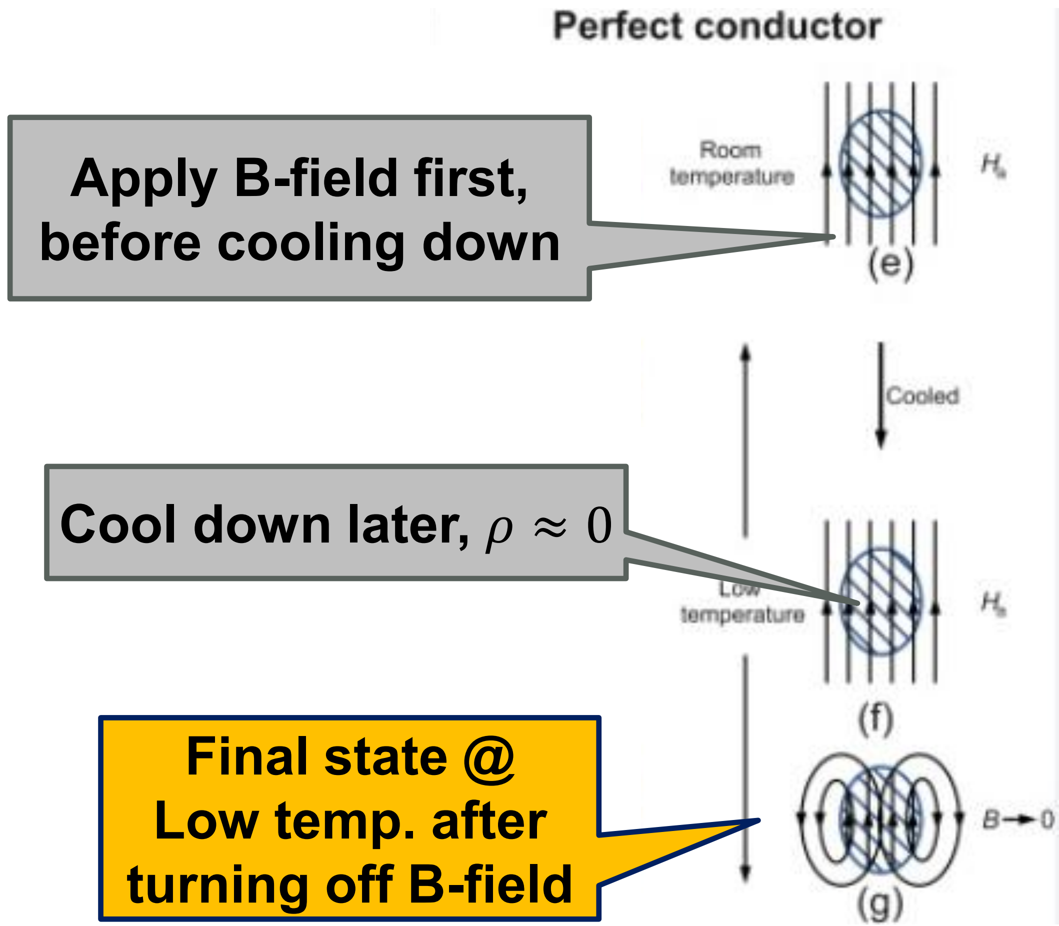
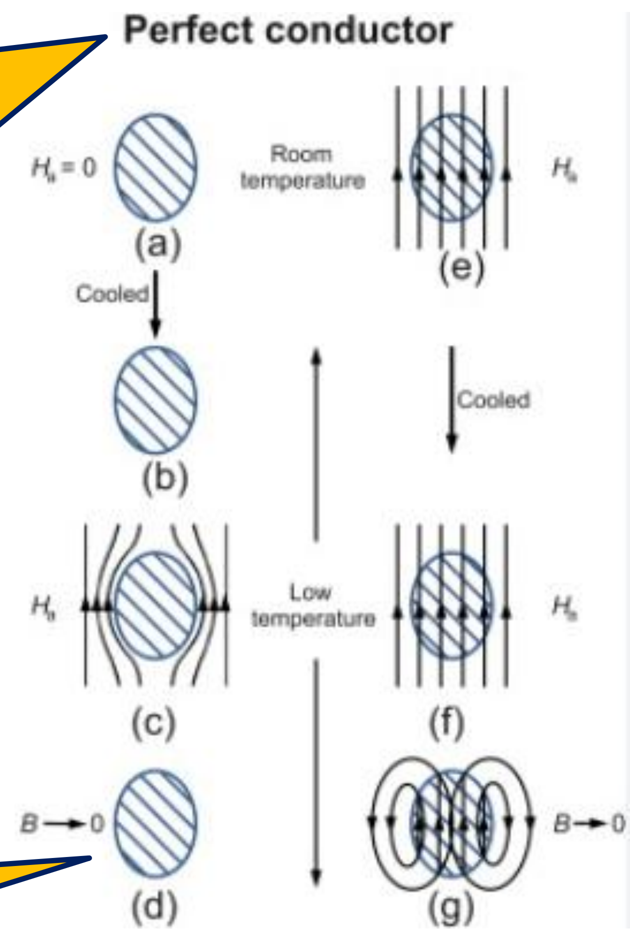


Route I:
superconductivity in historical order

- **Phenomenology**
  → Is a perfect conductor all about the superconductivity?

<u>Scenario B</u>: it's not a thermodynamic phase, but a good conductor
it should depend only on the present thermodynamic variables

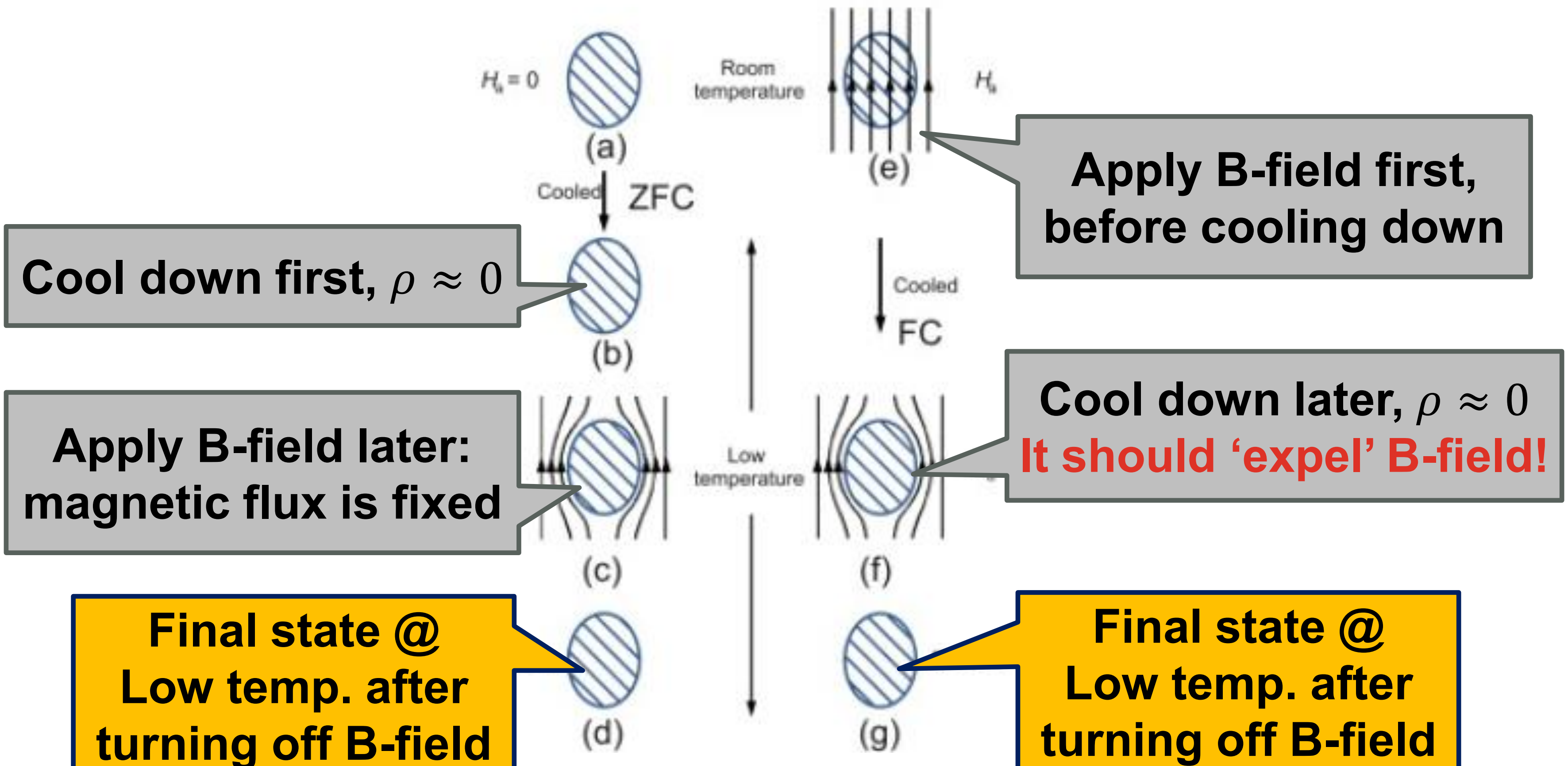


Different Final states @ Low temp. after turning off B-field

Route I: superconductivity in historical order

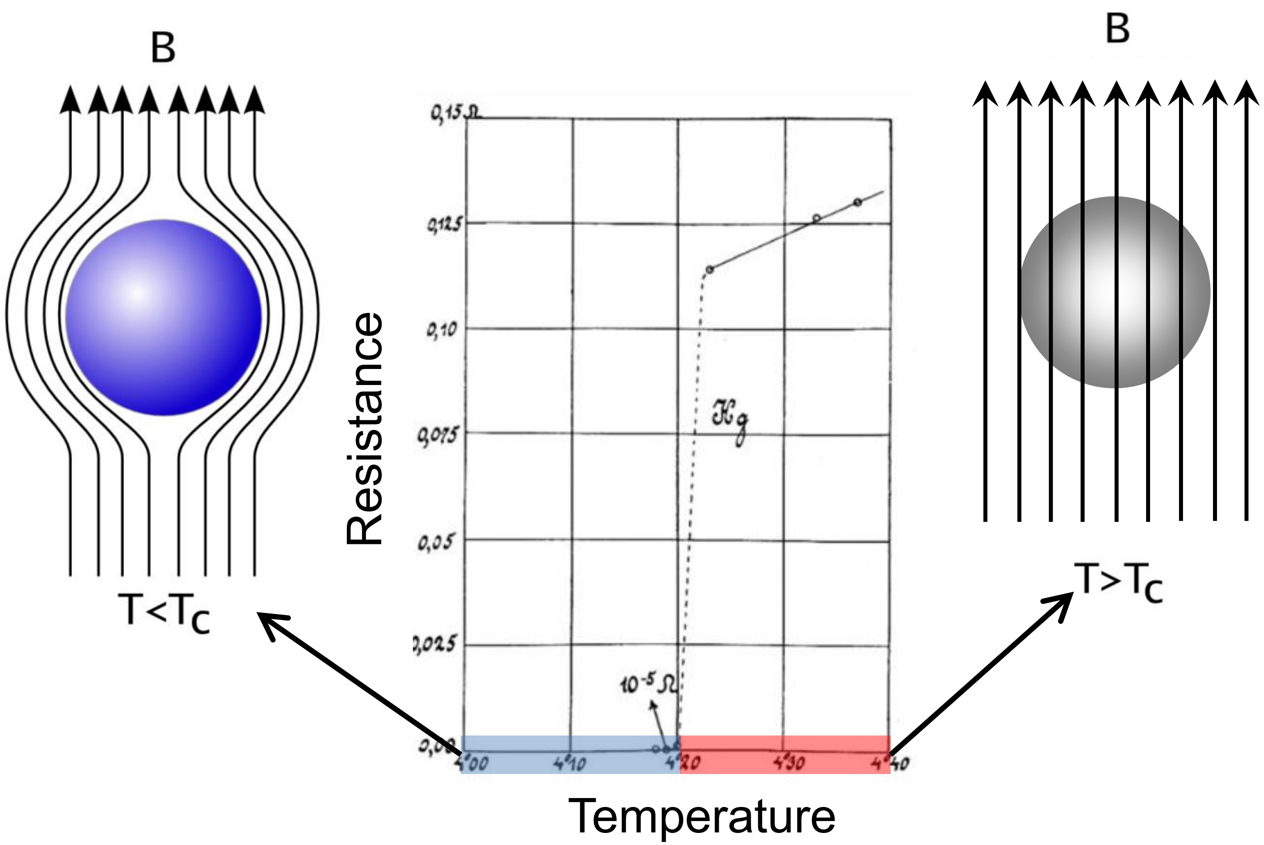
- **Phenomenology**
  → Is a perfect conductor all about the superconductivity?

<u>Scenario A</u>: it's a thermodynamic phase (superconductivity)



Route I: superconductivity in historical order

- **Phenomenology**
  → Superconductivity is a new thermodynamic phase
  = Perfect conductor/diamagnetism



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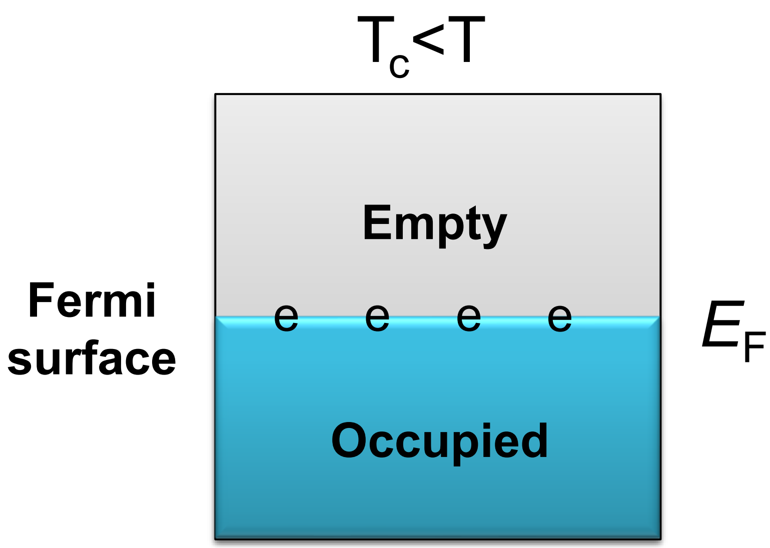
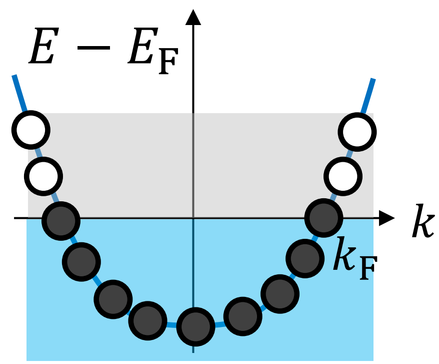
Route I: superconductivity in historical order

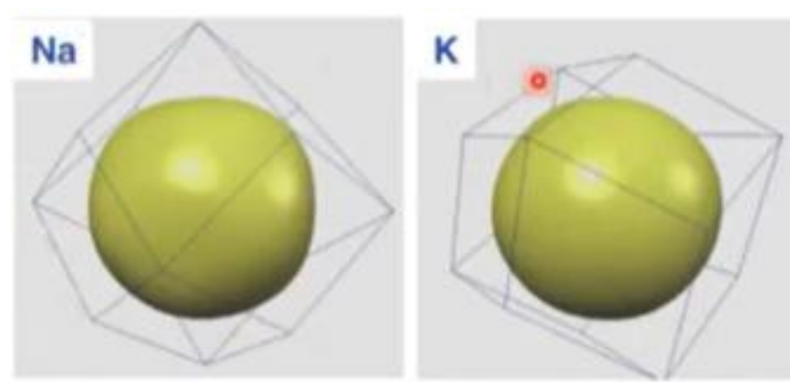
- **Phenomenology**
  → Superconductivity is a new thermodynamic phase
  = Perfect conductor/diamagnetism
- **New thermodynamic phase & its new ground state**
  → Condensation of Cooper pairs



Metal
$|\mathrm{GS}\rangle$ = Fermi Sea



$$|\mathrm{FS}\rangle = \prod_{k<k_\mathrm{F}} \hat{d}_k^\dagger |0\rangle$$



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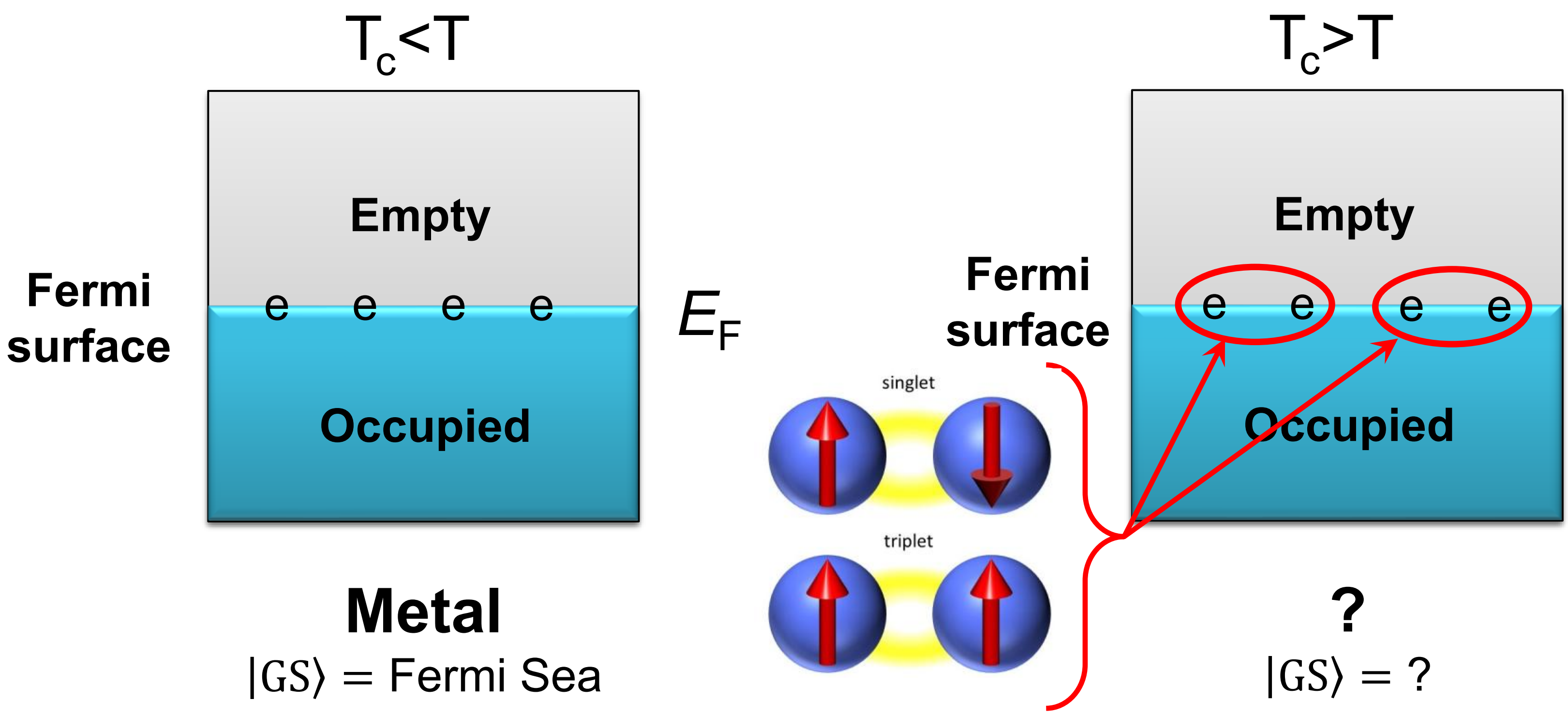

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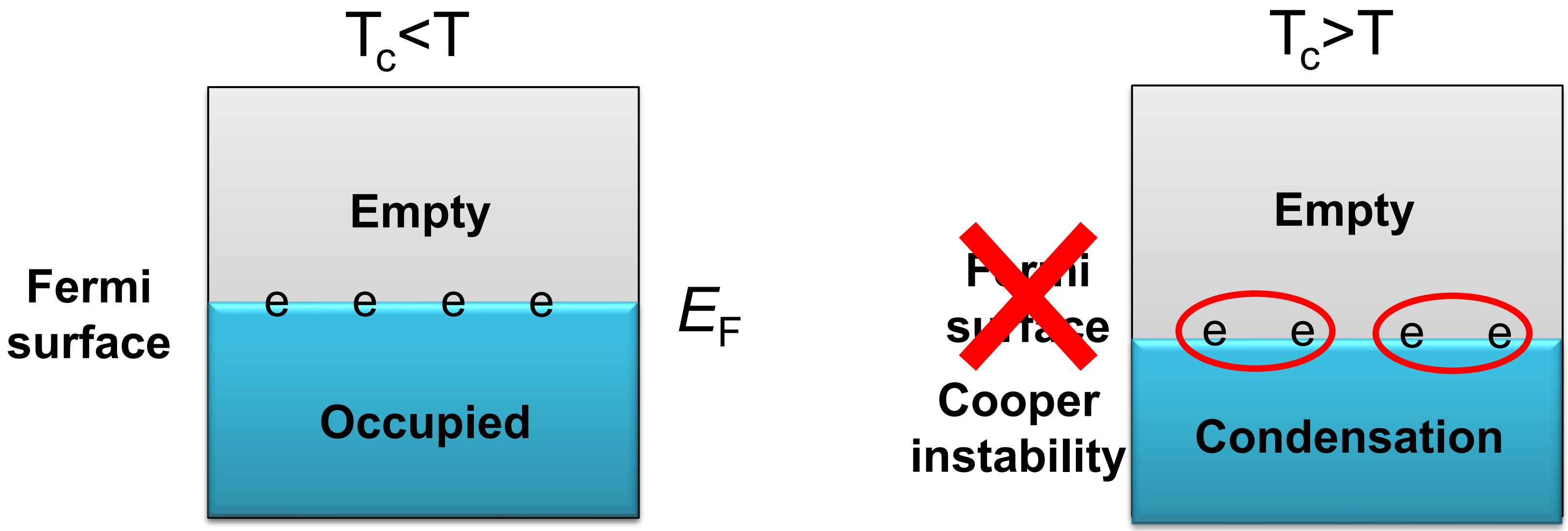

Metal	**?**
$|GS\rangle$ = Fermi Sea	$|GS\rangle = ?$

Route I: superconductivity in historical order

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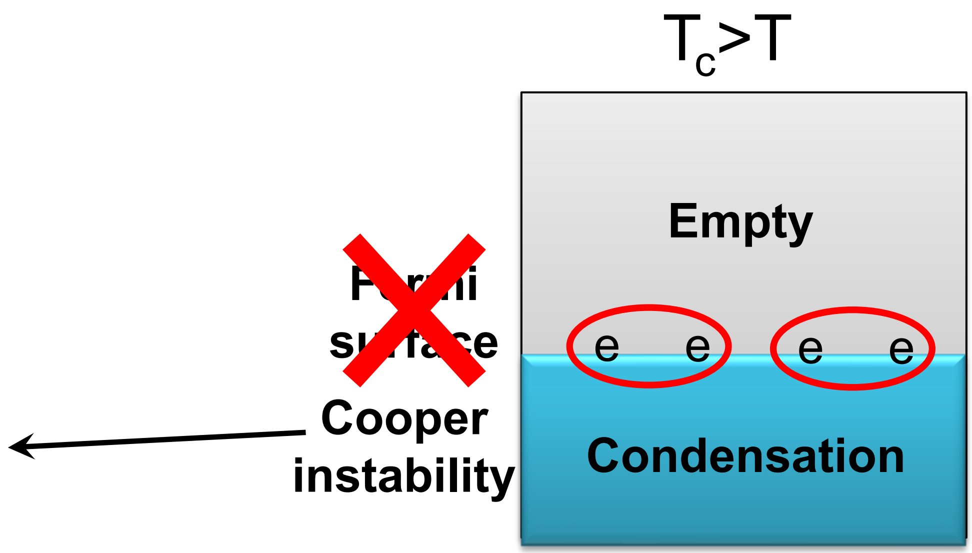

?

$|\mathrm{GS}\rangle = ?$

Route I:
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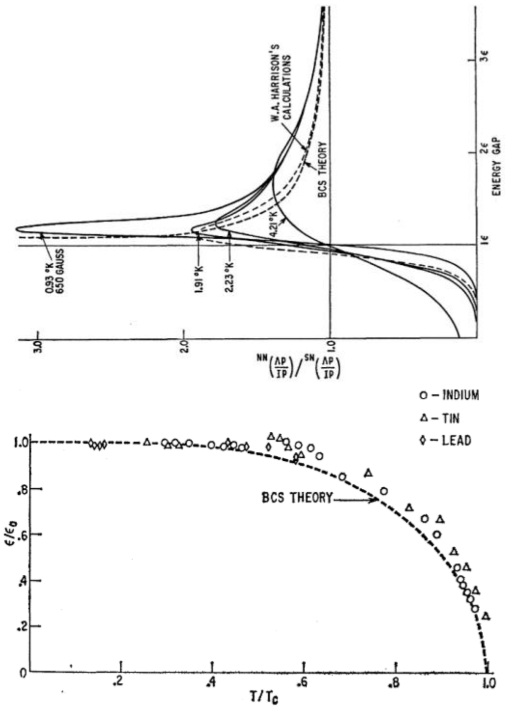
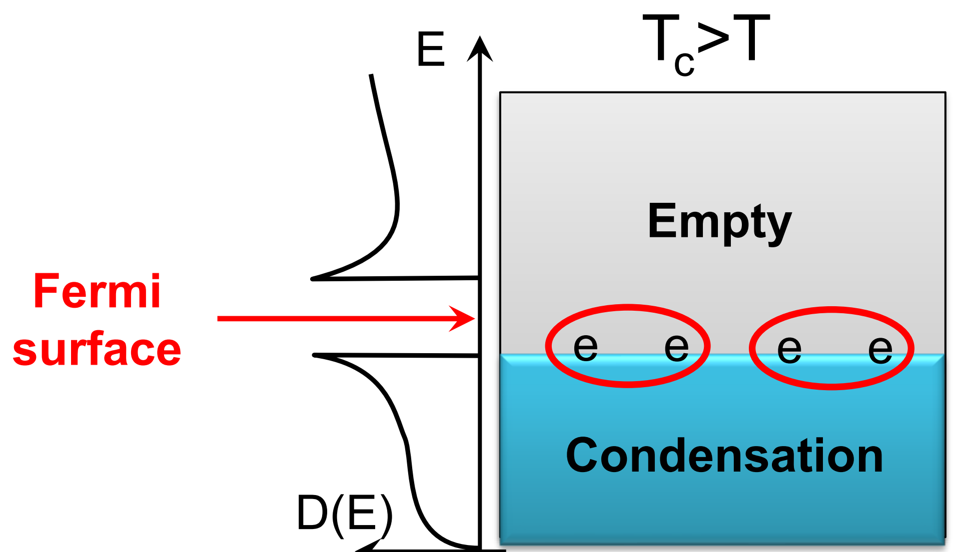


I. Giaever and K. Megerle, Phys. Rev. **122** 1101 (1961).



Superconductor

$|GS\rangle =$ Condensation of Copper pairs

Route I: superconductivity in historical order

- **Phenomenology**
  → Superconductivity is a new thermodynamic phase
  = Perfect conductor/diamagnetism
- **New thermodynamic phase & its**
  → Condensation of Cooper pairs
  → SC gap is a signature of the condensation

The SC gap should
(1) be pinned at E_F
(2) appear as lowering T

Route I: superconductivity in historical order

- **Phenomenology**
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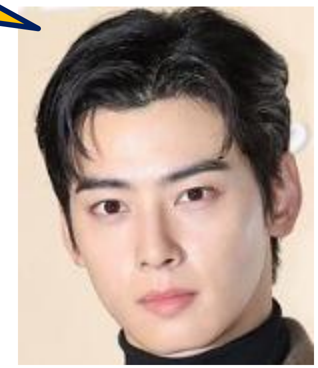
- **Physical mechanism for SC**
  → Phonon-mediated attractive interaction

Route I:
superconductivity in historical order

The earth cannot bear two suns

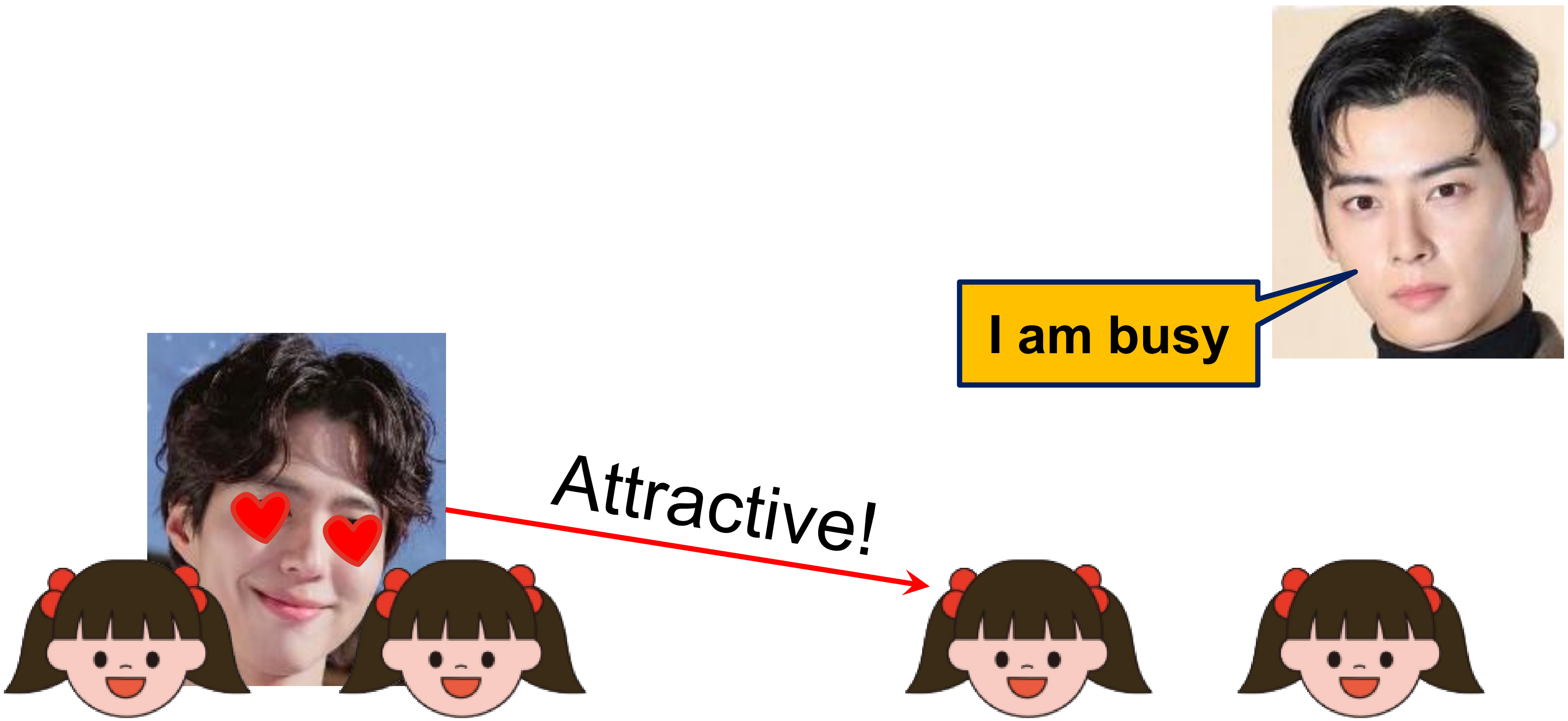
Repulsive!



Route I:
superconductivity in historical order

Neutral
in peace

Route I: superconductivity in historical order



Route I:
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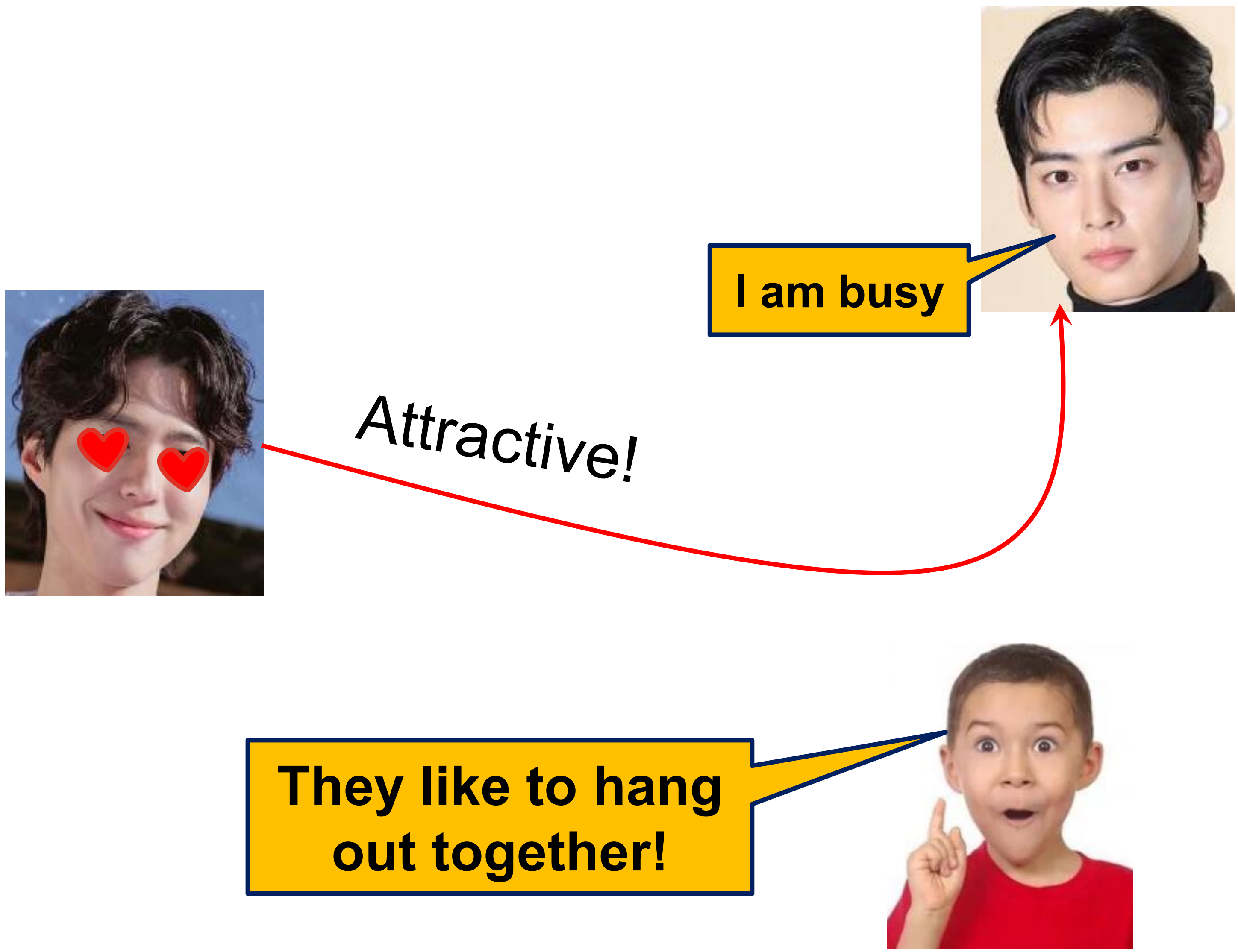

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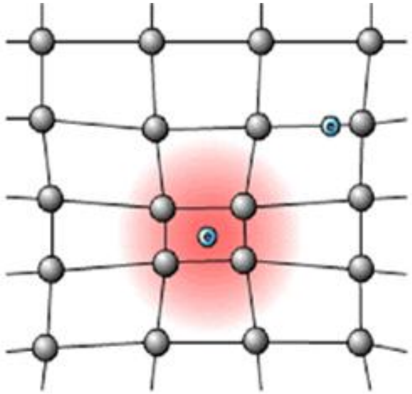
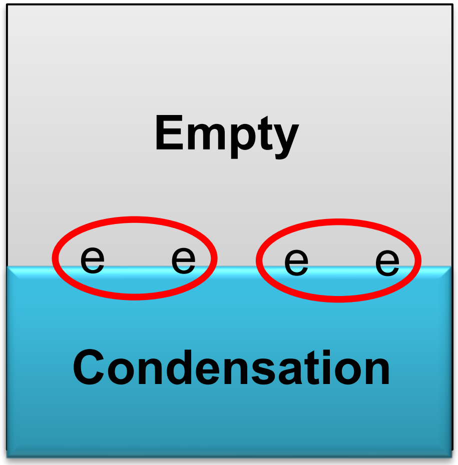


$T_c > T$



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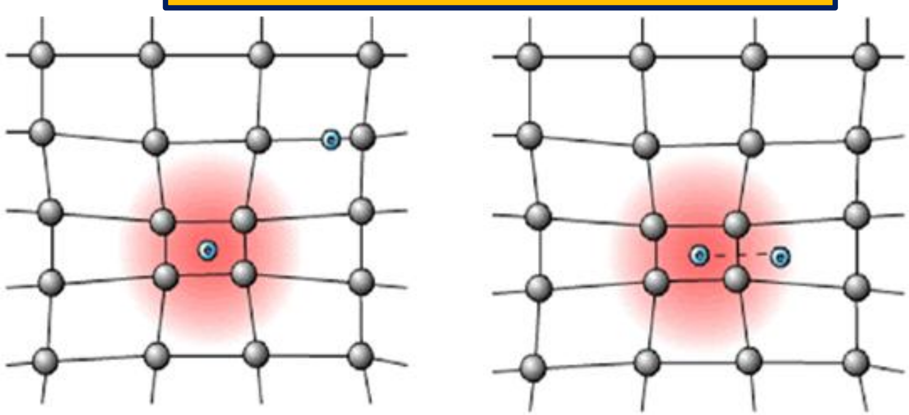
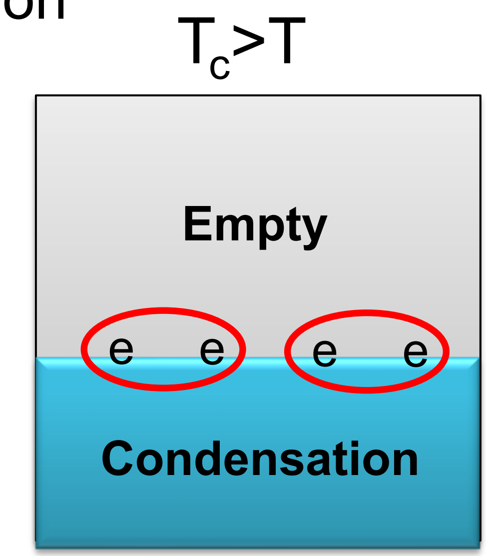
Ions move slow, while *electrons move fast*





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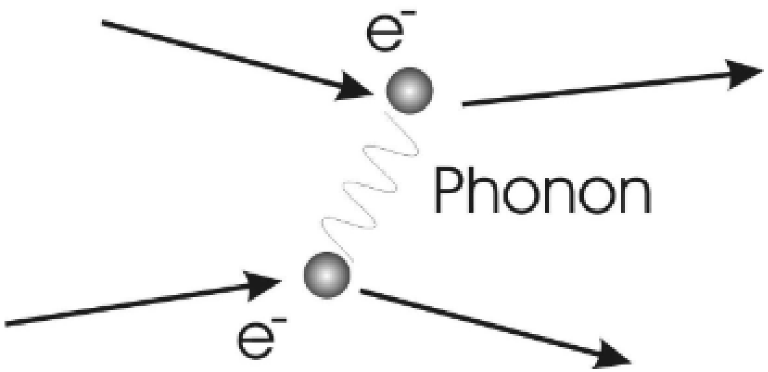

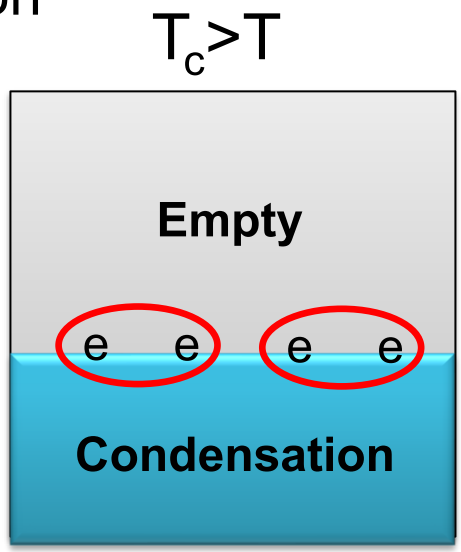

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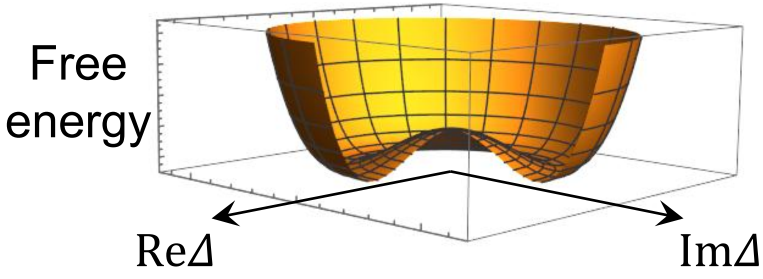
- **Spontaneous global U(1) symmetry breaking**
  → Particle number is uncertain and $[\hat{\Phi}, \hat{N}] \sim 1$

$T_c > T$



Order parameter,
$\langle \hat{d}_{k,\uparrow} \hat{d}_{-k,\downarrow} \rangle = \Delta e^{i\Phi}$

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- **Physical mechanism for SC**
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- **Spontaneous global U(1) symmetry breaking**
  → Particle number is uncertain and $[\hat{\Phi}, \hat{N}] \sim 1$
  → SC phase $\boldsymbol{\Phi}$ is a good dynamical variable

Macroscopic wavefunction for SCs

$$|\boldsymbol{\Phi}\rangle = |0\rangle + e^{i\boldsymbol{\Phi}}|2e\rangle + e^{2i\boldsymbol{\Phi}}|4e\rangle + \cdots$$

(unbreakable Cooper pairs, i.e., **all relevant energy scales << Δ**)

※ Recall $[\hat{x}, \hat{k}] \sim i$ and when k is a good q-#., $|k\rangle \propto \int e^{ikx}|x\rangle$

Route I: superconductivity in historical order

- **Theoretical description of superconductivity**

  → How to write ‘condensation of Cooper pairs’ mathematically

  → Coherent state with infinitely many Bosons

$$|\alpha\rangle \propto a_0|0\rangle + a_1|1\rangle + a_2|2\rangle + \cdots$$

  → But to occupy a single quantum number

$$|\alpha\rangle \propto \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

  or equivalently defined by $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

$$\hat{a}|\alpha\rangle$$
$$= \hat{a}(a_0|0\rangle + a_1|1\rangle + a_2|2\rangle + \cdots\ )$$
$$= 0 + a_1\sqrt{1}|0\rangle + a_2\sqrt{2}|1\rangle + \cdots$$
$$= \frac{\alpha^1}{\sqrt{1}}\sqrt{1}|0\rangle + \frac{\alpha^2}{\sqrt{1}\sqrt{2}}\sqrt{2}|1\rangle + \cdots$$
$$= \alpha(|0\rangle + \alpha|1\rangle + \cdots\ ) = \alpha|\alpha\rangle$$

Macroscopic wavefunction remains the same for extracting a single particle

Route I: superconductivity in historical order

- **Theoretical description of superconductivity**
  → How to write ‘condensation of Cooper pairs’ mathematically
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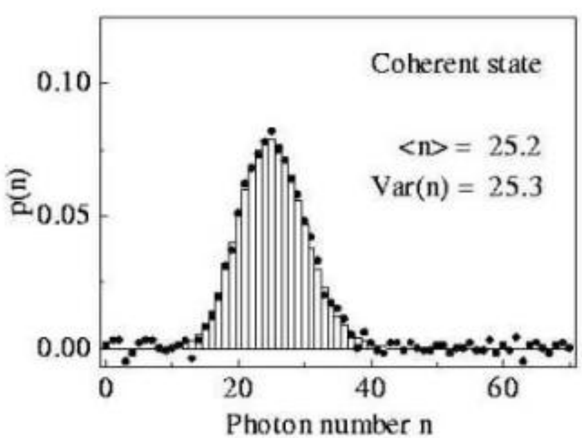
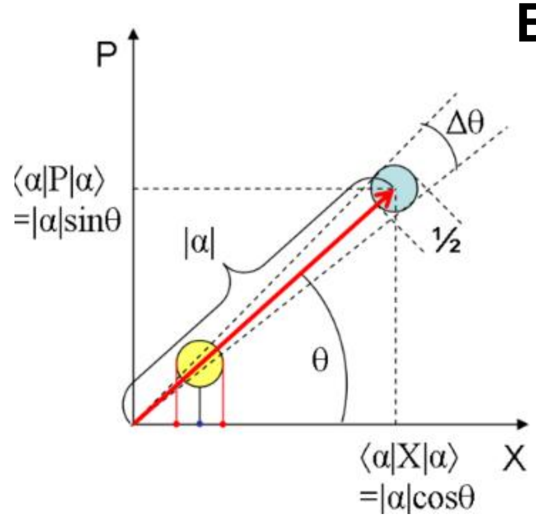
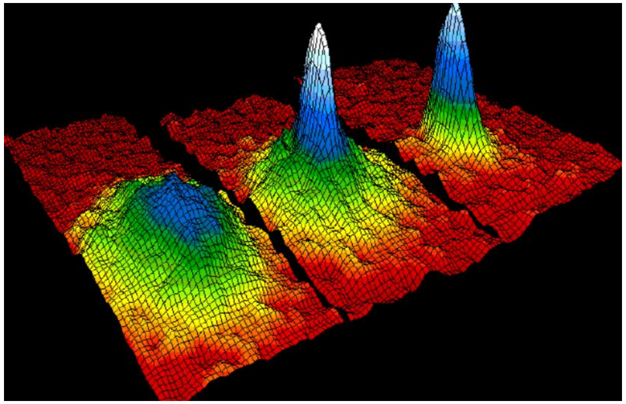
  or equivalently defined by $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$
  → if N-1=N, N=? Then what about N+1=?





Bose-Einstein Condensate



Route I: superconductivity in historical order

- **What Bardeen-Cooper-Schrieffer (BCS) tried**
  → They wrote down a (trial) BCS wavefunction
  as a coherent state of Cooper pairs

Coherent state of a boson

$$|\alpha\rangle \propto \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle = e^{\alpha \hat{a}^\dagger} |0\rangle$$

Recall the undergraduate QM

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$
$$\hat{a}^\dagger |0\rangle = \sqrt{1} |1\rangle$$
$$\hat{a}^\dagger |1\rangle = \sqrt{2} |2\rangle$$
$$\hat{a}^\dagger |2\rangle = \sqrt{3} |3\rangle$$
$$\hat{a}^\dagger |3\rangle = \sqrt{4} |4\rangle$$
$$\vdots$$

$$|3\rangle = \frac{\hat{a}^\dagger}{\sqrt{3}} |2\rangle = \frac{\hat{a}^\dagger}{\sqrt{3}} \frac{\hat{a}^\dagger}{\sqrt{2}} |1\rangle = \frac{(\hat{a}^\dagger)^3}{\sqrt{3!}} |0\rangle$$

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle$$

Route I: superconductivity in historical order

- **What Bardeen-Cooper-Schrieffer (BCS) tried**
  → They wrote down a (trial) BCS wavefunction as a coherent state of Cooper pairs

Coherent state of a Cooper pair

Use $\hat{P}^\dagger_{\boldsymbol{k}} = \hat{d}^\dagger_{\boldsymbol{k},\uparrow}\hat{d}^\dagger_{-\boldsymbol{k},\downarrow}$ instead $\hat{a}^\dagger$

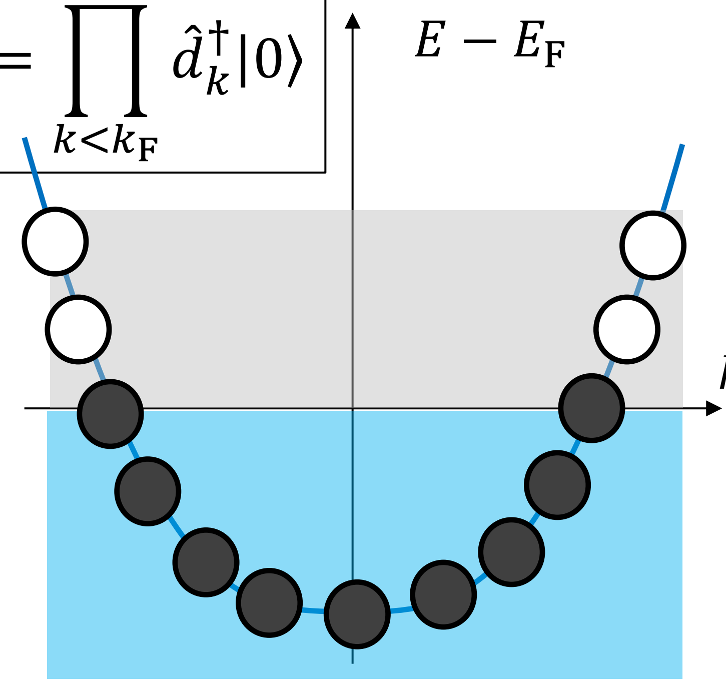
$$e^{\alpha_{\boldsymbol{k}}\hat{P}^\dagger_{\boldsymbol{k}}}|0\rangle$$

$$= \left[1 + \alpha_{\boldsymbol{k}}\hat{d}^\dagger_{\boldsymbol{k},\uparrow}\hat{d}^\dagger_{-\boldsymbol{k},\downarrow} + \frac{\left(\alpha_{\boldsymbol{k}}\hat{d}^\dagger_{\boldsymbol{k},\uparrow}\hat{d}^\dagger_{-\boldsymbol{k},\downarrow}\right)^2}{\sqrt{2}} + \cdots\right]|0\rangle$$

$$= \left(1 + \alpha_{\boldsymbol{k}}\hat{d}^\dagger_{\boldsymbol{k},\uparrow}\hat{d}^\dagger_{-\boldsymbol{k},\downarrow}\right)|0\rangle$$

$$|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}}\left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}}\hat{d}^\dagger_{\boldsymbol{k},\uparrow}\hat{d}^\dagger_{-\boldsymbol{k},\downarrow}\right)|0\rangle$$

$$|\mathrm{FS}\rangle = \prod_{k<k_\mathrm{F}} \hat{d}^\dagger_k|0\rangle$$



Fermi Sea

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$$e^{\alpha_{\boldsymbol{k}}\hat{P}^\dagger_{\boldsymbol{k}}}|0\rangle$$

$$= \left[1 + \alpha_{\boldsymbol{k}}\hat{d}^\dagger_{\boldsymbol{k},\uparrow} \cdots \right]|0\rangle$$

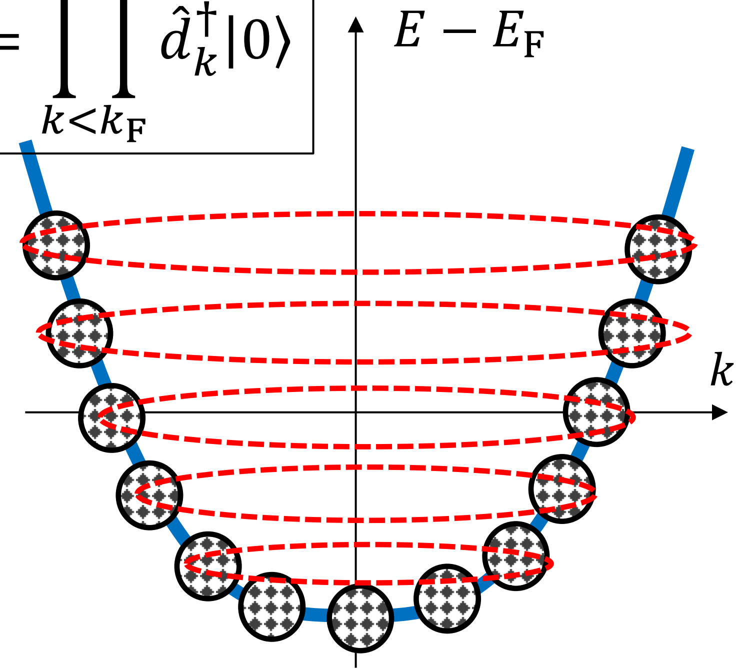
$$= (1 + \alpha_{\boldsymbol{k}}\hat{d}^\dagger_{\boldsymbol{k},\uparrow}\hat{d}^\dagger_{-\boldsymbol{k},\downarrow})|0\rangle$$

$\alpha_{\boldsymbol{k}}$'s are unknowns

$$|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}}(u_{\boldsymbol{k}} + v_{\boldsymbol{k}}\hat{d}^\dagger_{\boldsymbol{k},\uparrow}\hat{d}^\dagger_{-\boldsymbol{k},\downarrow})|0\rangle$$

$u_{\boldsymbol{k}}$, $v_{\boldsymbol{k}}$ are unknowns

$$|\mathrm{FS}\rangle = \prod_{k<k_\mathrm{F}} \hat{d}^\dagger_k|0\rangle$$



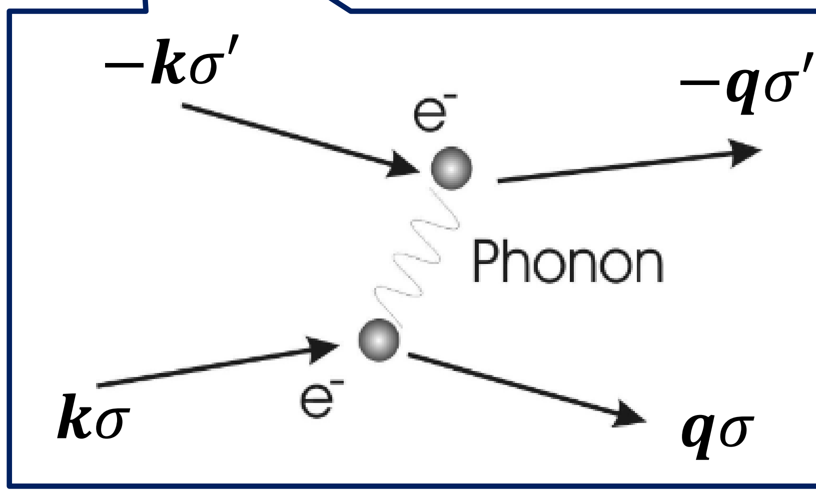
BCS GS

Route I: superconductivity in historical order

- **What Bardeen-Cooper-Schrieffer (BCS) tried**
  → They wrote down a (trial) BCS wavefunction
  as a coherent state of Cooper pairs
  → Apply variational method (as we have a trial wavefunction!)

$$\hat{H} = \sum_{\boldsymbol{k},\sigma=\uparrow,\downarrow} \xi_k \hat{d}^\dagger_{\boldsymbol{k}\sigma} \hat{d}_{\boldsymbol{k}\sigma} + \frac{1}{2} \sum_{\boldsymbol{k},\boldsymbol{q},\sigma,\sigma'} V(\boldsymbol{k}-\boldsymbol{q}) \hat{d}^\dagger_{\boldsymbol{k}\sigma} \hat{d}^\dagger_{-\boldsymbol{q}\sigma'} \hat{d}_{-\boldsymbol{k}\sigma'} \hat{d}_{\boldsymbol{k}\sigma}$$

$\xi_k = \epsilon_k - E_\mathrm{F}$,
e.g., $\xi_k = \frac{\hbar^2 k^2}{2m} - E_\mathrm{F}$



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$$\hat{H} = \sum_{\boldsymbol{k},\sigma=\uparrow,\downarrow} \xi_k \hat{d}^\dagger_{\boldsymbol{k}\sigma} \hat{d}_{\boldsymbol{k}\sigma} + \frac{1}{2} \sum_{\boldsymbol{k},\boldsymbol{q},\sigma,\sigma'} V(\boldsymbol{k}-\boldsymbol{q}) \hat{d}^\dagger_{\boldsymbol{k}\sigma} \hat{d}^\dagger_{-\boldsymbol{q}\sigma'} \hat{d}_{-\boldsymbol{k}\sigma'} \hat{d}_{\boldsymbol{k}\sigma}$$

Now, minimize $E = \langle \mathrm{BCS} | \hat{H} | \mathrm{BCS} \rangle$

$$E = 2\sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}} |v_{\boldsymbol{k}}|^2 + \sum_{\boldsymbol{k},\boldsymbol{q}} V u_{\boldsymbol{q}} v^*_{\boldsymbol{q}} u^*_{\boldsymbol{k}} v_{\boldsymbol{k}}$$

$$= 2\sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}} |v_{\boldsymbol{k}}|^2 + \sum_{\boldsymbol{k},\boldsymbol{q}} V |u_{\boldsymbol{q}}||v_{\boldsymbol{q}}||u_{\boldsymbol{k}}||v_{\boldsymbol{k}}| e^{i(\varphi_{\boldsymbol{k}}-\varphi_{\boldsymbol{q}})}$$

$$= 2\sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}} \cos^2\theta_{\boldsymbol{k}} + \sum_{\boldsymbol{k},\boldsymbol{q}} V \sin\theta_{\boldsymbol{q}} \cos\theta_{\boldsymbol{q}} \sin\theta_{\boldsymbol{k}} \cos\theta_{\boldsymbol{k}}\, e^{i(\varphi_{\boldsymbol{k}}-\varphi_{\boldsymbol{q}})}$$

$$= 2\sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}} \cos^2\theta_{\boldsymbol{k}} + \frac{1}{4}\sum_{\boldsymbol{k},\boldsymbol{q}} V \sin 2\theta_{\boldsymbol{q}} \sin 2\theta_{\boldsymbol{k}}\, e^{i(\varphi_{\boldsymbol{k}}-\varphi_{\boldsymbol{q}})}$$

$$\frac{\delta E}{\delta \varphi_{\boldsymbol{p}}} = 0 \;\; \& \;\; \frac{\delta E}{\delta \theta_{\boldsymbol{p}}} = 0$$

$\varphi_{\boldsymbol{k}} = \arg\{u^*_{\boldsymbol{k}} v_{\boldsymbol{k}}\}$

$|u_{\boldsymbol{k}}|^2 + |v_{\boldsymbol{k}}|^2 = 1$

$|u_{\boldsymbol{k}}| = \sin\theta_{\boldsymbol{k}}$
$\& \; |v_{\boldsymbol{k}}| = \cos\theta_{\boldsymbol{k}}$
$(0 \le \theta_{\boldsymbol{k}} \le \frac{\pi}{2})$

Route I: superconductivity in historical order

- **What Bardeen-Cooper-Schrieffer (BCS) tried**
  → They wrote down a (trial) BCS wavefunction
  as a coherent state of Cooper pairs
  → Apply variational method (as we have a trial wavefunction!)

$$\boxed{\frac{\delta E}{\delta \varphi_{\boldsymbol{p}}} = 0} \Leftrightarrow 0 = \frac{\delta}{\delta \varphi_{\boldsymbol{p}}}\left[\sum_{\boldsymbol{k},\boldsymbol{q}} V(\boldsymbol{k}-\boldsymbol{q})|u_{\boldsymbol{q}}||v_{\boldsymbol{q}}||u_{\boldsymbol{k}}||v_{\boldsymbol{k}}|\, e^{i(\varphi_{\boldsymbol{k}}-\varphi_{\boldsymbol{q}})}\right]$$

$$= \sum_{\boldsymbol{k},\boldsymbol{q}} \underbrace{V(\boldsymbol{k}-\boldsymbol{q})|u_{\boldsymbol{q}}||v_{\boldsymbol{q}}||u_{\boldsymbol{k}}||v_{\boldsymbol{k}}|}\, \frac{\delta}{\delta \varphi_{\boldsymbol{p}}} e^{i(\varphi_{\boldsymbol{k}}-\varphi_{\boldsymbol{q}})}$$

$$= \sum_{\boldsymbol{k},\boldsymbol{q}} \quad e^{i(\varphi_{\boldsymbol{k}}-\varphi_{\boldsymbol{q}})}\left(i\frac{\delta \varphi_{\boldsymbol{k}}}{\delta \varphi_{\boldsymbol{p}}} - i\frac{\delta \varphi_{\boldsymbol{q}}}{\delta \varphi_{\boldsymbol{p}}}\right) = i\sum_{\boldsymbol{k},\boldsymbol{q}} \quad e^{i(\varphi_{\boldsymbol{k}}-\varphi_{\boldsymbol{q}})}\left(\delta_{\boldsymbol{p},\boldsymbol{k}} - \delta_{\boldsymbol{p},\boldsymbol{q}}\right)$$

$$= i\sum_{\boldsymbol{q}} V(\boldsymbol{p}-\boldsymbol{q})|u_{\boldsymbol{q}}||v_{\boldsymbol{q}}||u_{\boldsymbol{p}}||v_{\boldsymbol{p}}|e^{i(\varphi_{\boldsymbol{p}}-\varphi_{\boldsymbol{q}})}$$

$$-i\sum_{\boldsymbol{k}} V(\boldsymbol{k}-\boldsymbol{p})|u_{\boldsymbol{p}}||v_{\boldsymbol{p}}||u_{\boldsymbol{k}}||v_{\boldsymbol{k}}|e^{i(\varphi_{\boldsymbol{k}}-\varphi_{\boldsymbol{p}})}$$

$$= -2\sum_{\boldsymbol{q}} V(\boldsymbol{p}-\boldsymbol{q})|u_{\boldsymbol{q}}||v_{\boldsymbol{q}}||u_{\boldsymbol{p}}||v_{\boldsymbol{p}}| \sin(\varphi_{\boldsymbol{p}}-\varphi_{\boldsymbol{q}})$$

$$\boxed{0 = \sum_{\boldsymbol{q}} V(\boldsymbol{p}-\boldsymbol{q}) \sin 2\theta_{\boldsymbol{p}} \sin 2\theta_{\boldsymbol{q}} \sin(\varphi_{\boldsymbol{p}}-\varphi_{\boldsymbol{q}})}$$

Route I: superconductivity in historical order

- **What Bardeen-Cooper-Schrieffer (BCS) tried**
  → They wrote down a (trial) BCS wavefunction
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  → Apply variational method (as we have a trial wavefunction!)

$$\frac{\delta E}{\delta \varphi_{\boldsymbol{p}}} = 0 \Leftrightarrow \sin 2\theta_{\boldsymbol{p}} \sum_{\boldsymbol{q}} V(\boldsymbol{p}-\boldsymbol{q}) \sin 2\theta_{\boldsymbol{q}} \sin(\varphi_{\boldsymbol{p}} - \varphi_{\boldsymbol{q}}) = 0$$

$|u_{\boldsymbol{k}}| = \sin\theta_{\boldsymbol{k}}$ & $|v_{\boldsymbol{k}}| = \cos\theta_{\boldsymbol{k}}$
$(0 \leq \theta_{\boldsymbol{k}} \leq \frac{\pi}{2})$
$V(\boldsymbol{p}-\boldsymbol{q}) \sin 2\theta_{\boldsymbol{q}} > 0$ or
$V(\boldsymbol{p}-\boldsymbol{q}) \sin 2\theta_{\boldsymbol{q}} < 0$ for all $\boldsymbol{p}, \boldsymbol{q}$

Hence, $\sin(\varphi_{\boldsymbol{p}} - \varphi_{\boldsymbol{q}}) = 0$
for all $\boldsymbol{q}$'s for a given $\boldsymbol{p}$, i.e.,

$$\varphi_{\boldsymbol{p}} - \varphi_{\boldsymbol{q}} = 0$$

Note: $\varphi_{\boldsymbol{p}} - \varphi_{\boldsymbol{q}} = \pi$ is not an option due to single-valuedness of $\varphi_{\boldsymbol{p}}$. (Assume $\varphi_{\boldsymbol{p}} = \varphi_{\boldsymbol{q}} + \pi$. This yields $\varphi_{\boldsymbol{p}} \neq \varphi_{\boldsymbol{p}}$ if $\boldsymbol{q} = \boldsymbol{p}$)

Route I: superconductivity in historical order

- **What Bardeen-Cooper-Schrieffer (BCS) tried**
  $\longrightarrow$ They wrote down a (trial) BCS wavefunction
  as a coherent state of Cooper pairs
  $\longrightarrow$ Apply variational method (as we have a trial wavefunction!)

$$\frac{\delta E}{\delta \varphi_{\boldsymbol{p}}} = 0 \Leftrightarrow \sin 2\theta_{\boldsymbol{p}} \sum_{\boldsymbol{q}} V(\boldsymbol{p}-\boldsymbol{q}) \sin 2\theta_{\boldsymbol{q}} \sin\left(\varphi_{\boldsymbol{p}} - \varphi_{\boldsymbol{q}}\right) = 0$$

Hence, $\sin\left(\varphi_{\boldsymbol{p}} - \varphi_{\boldsymbol{q}}\right) = 0$ for all $\boldsymbol{q}$'s for a given $\boldsymbol{p}$, i.e.,

$$\varphi_{\boldsymbol{p}} - \varphi_{\boldsymbol{q}} = 0$$

Note: $\varphi_{\boldsymbol{p}} - \varphi_{\boldsymbol{q}} = \pi$ is not an option due to single-valuedness of $\varphi_{\boldsymbol{p}}$. (Assume $\varphi_{\boldsymbol{p}} = \varphi_{\boldsymbol{q}} + \pi$. This yields $\varphi_{\boldsymbol{p}} \neq \varphi_{\boldsymbol{p}}$ if $\boldsymbol{q} = \boldsymbol{p}$)

All electrons at $\boldsymbol{k}$'s share an identical global phase φ, which can be arbitrary.

$$\begin{aligned} |\mathrm{BCS}\rangle &= \prod_{\boldsymbol{k}} \left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}} \hat{d}^{\dagger}_{\boldsymbol{k},\uparrow} \hat{d}^{\dagger}_{-\boldsymbol{k},\downarrow}\right)|0\rangle \\ &= \prod_{\boldsymbol{k}} \left(|u_{\boldsymbol{k}}| + |v_{\boldsymbol{k}}| e^{i\varphi} \hat{d}^{\dagger}_{\boldsymbol{k},\uparrow} \hat{d}^{\dagger}_{-\boldsymbol{k},\downarrow}\right)|0\rangle \\ &= \prod_{\boldsymbol{k}} \left(|u_{\boldsymbol{k}}| + |v_{\boldsymbol{k}}| \hat{d}^{\dagger}_{\boldsymbol{k},\uparrow} \hat{d}^{\dagger}_{-\boldsymbol{k},\downarrow}\right)|0\rangle \end{aligned}$$

(by doing $e^{i\varphi/2} \hat{d}^{\dagger}_{\boldsymbol{k},\sigma} \mapsto \hat{d}^{\dagger}_{\boldsymbol{k},\sigma}$)

Route I: superconductivity in historical order

- **What Bardeen-Cooper-Schrieffer (BCS) tried**
  → They wrote down a (trial) BCS wavefunction
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$$\frac{\delta E}{\delta \theta_{\boldsymbol{p}}} = 0 \Leftrightarrow -2\xi_{\boldsymbol{p}} \sin 2\theta_{\boldsymbol{p}} + \cos 2\theta_{\boldsymbol{p}} \sum_{\boldsymbol{q}} V(\boldsymbol{p}-\boldsymbol{q}) \sin 2\theta_{\boldsymbol{q}} = 0$$

$$\Leftrightarrow \tan 2\theta_{\boldsymbol{p}} = \sum_{\boldsymbol{q}} \frac{V(\boldsymbol{p}-\boldsymbol{q}) \sin 2\theta_{\boldsymbol{q}}}{2\xi_{\boldsymbol{p}}} = -\frac{\Delta_{\boldsymbol{p}}}{\xi_{\boldsymbol{p}}}$$

$$\Leftrightarrow \Delta_{\boldsymbol{p}} \equiv -\frac{1}{2}\sum_{\boldsymbol{q}} V(\boldsymbol{p}-\boldsymbol{q}) \sin 2\theta_{\boldsymbol{q}} = -\frac{1}{2}\sum_{\boldsymbol{q}} V(\boldsymbol{k}-\boldsymbol{q}) \sqrt{\frac{\tan^2 2\theta_{\boldsymbol{q}}}{1+\tan^2 2\theta_{\boldsymbol{q}}}}$$

$$= -\frac{1}{2}\sum_{\boldsymbol{q}} V(\boldsymbol{k}-\boldsymbol{q}) \frac{\Delta_{\boldsymbol{q}}/|\xi_{\boldsymbol{q}}|}{\sqrt{1+(\Delta_{\boldsymbol{q}}/\xi_{\boldsymbol{q}})^2}} \quad \text{(using } \tan 2\theta_{\boldsymbol{p}} = -\frac{\Delta_{\boldsymbol{p}}}{\xi_{\boldsymbol{p}}}\text{)}$$

$$\Delta_{\boldsymbol{p}} = -\frac{1}{2}\sum_{\boldsymbol{q}} V(\boldsymbol{p}-\boldsymbol{q}) \frac{\Delta_{\boldsymbol{q}}}{\sqrt{\xi_{\boldsymbol{q}}^2 + \Delta_{\boldsymbol{q}}^2}}$$

BCS gap equation at T=0

Route I: superconductivity in historical order

- **What Bardeen-Cooper-Schrieffer (BCS) tried**
  → They wrote down a (trial) BCS wavefunction as a coherent state of Cooper pairs
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$$\Delta_{\boldsymbol{k}} = -\frac{1}{2}\sum_{\boldsymbol{q}} V(\boldsymbol{k}-\boldsymbol{q})\,\Delta_{\boldsymbol{q}} \Big/ \sqrt{\xi_{\boldsymbol{q}}^2 + \Delta_{\boldsymbol{q}}^2}$$

Zero temp. result!

Let us examine a little bit more about $\Delta_{\boldsymbol{k}} \equiv -\frac{1}{2}\sum_{\boldsymbol{q}} V(\boldsymbol{k}-\boldsymbol{q}) \sin 2\theta_{\boldsymbol{q}}$

$$\Delta_{\boldsymbol{k}} = -\frac{1}{2}\sum_{\boldsymbol{q}} V(\boldsymbol{k}-\boldsymbol{q}) \frac{\Delta_{\boldsymbol{q}}}{E_{\boldsymbol{k}}} \Leftrightarrow \sin 2\theta_{\boldsymbol{k}} = \frac{\Delta_{\boldsymbol{k}}}{E_{\boldsymbol{k}}} \; \& \; \cos 2\theta_{\boldsymbol{k}} = -\frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}$$

$$u_{\boldsymbol{k}}^2 = \sin^2\theta_{\boldsymbol{k}} = \frac{1}{2}(1-\cos 2\theta_{\boldsymbol{k}}) = \frac{1}{2}\left(1+\frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right)$$

$$v_{\boldsymbol{k}}^2 = \cos^2\theta_{\boldsymbol{k}} = \frac{1}{2}(1+\cos 2\theta_{\boldsymbol{k}}) = \frac{1}{2}\left(1-\frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right)$$

$$\tan 2\theta_{\boldsymbol{p}} = -\frac{\Delta_{\boldsymbol{p}}}{\xi_{\boldsymbol{p}}}$$

Finally,

$$|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}} \left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}} \hat{d}_{\boldsymbol{k},\uparrow}^{\dagger} \hat{d}_{-\boldsymbol{k},\downarrow}^{\dagger}\right)|0\rangle$$

is determined.

Route I: superconductivity in historical order

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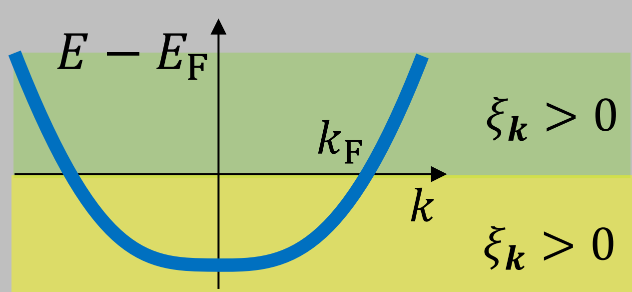
$$\Delta_{\boldsymbol{k}} = -\frac{1}{2}\sum_{\boldsymbol{q}} V(\boldsymbol{k}-\boldsymbol{q})\,\Delta_{\boldsymbol{q}} \Big/ \sqrt{\xi_{\boldsymbol{q}}^2 + \Delta_{\boldsymbol{q}}^2}$$

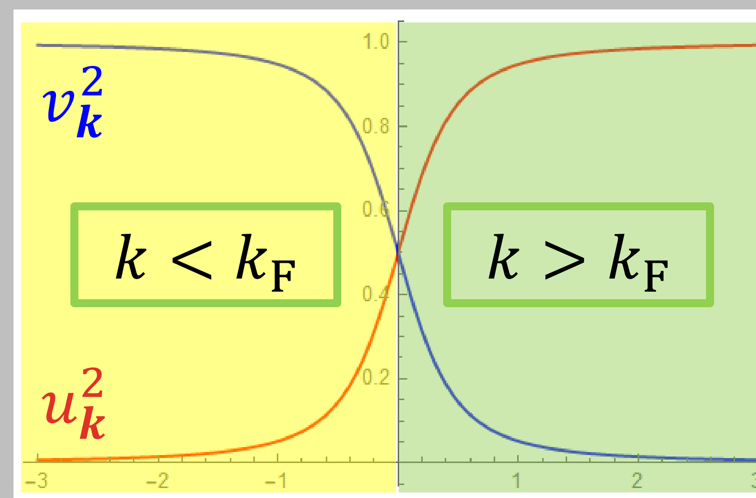
Zero temp. result!

Let us examine more about $|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}} \left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}} \hat{d}^\dagger_{\boldsymbol{k},\uparrow} \hat{d}^\dagger_{-\boldsymbol{k},\downarrow}\right)|0\rangle$

$$u_{\boldsymbol{k}}^2 = \sin^2\theta_{\boldsymbol{k}} = \frac{1}{2}(1-\cos 2\theta_{\boldsymbol{k}}) = \frac{1}{2}\left(1+\frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right)$$

$$v_{\boldsymbol{k}}^2 = \cos^2\theta_{\boldsymbol{k}} = \frac{1}{2}(1+\cos 2\theta_{\boldsymbol{k}}) = \frac{1}{2}\left(1-\frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right)$$





$\xi_{\boldsymbol{k}}$

Route I: superconductivity in historical order

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  → They wrote down a (trial) BCS wavefunction
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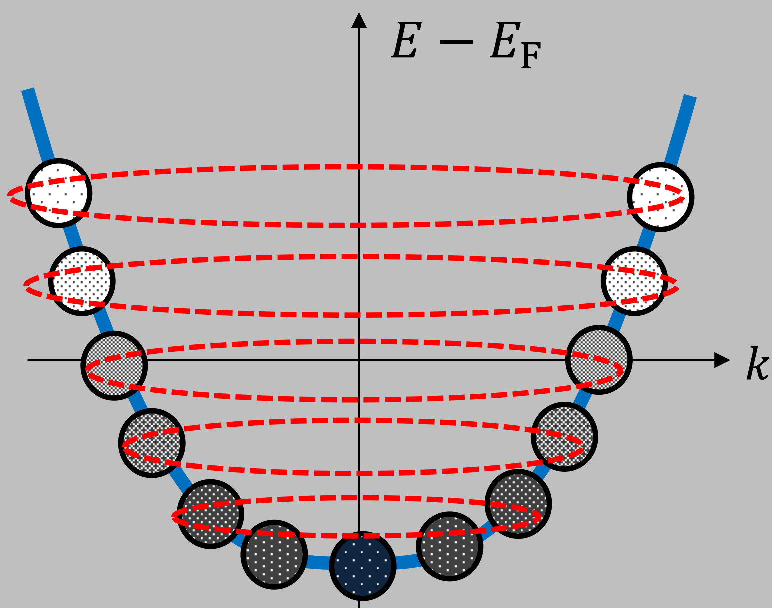
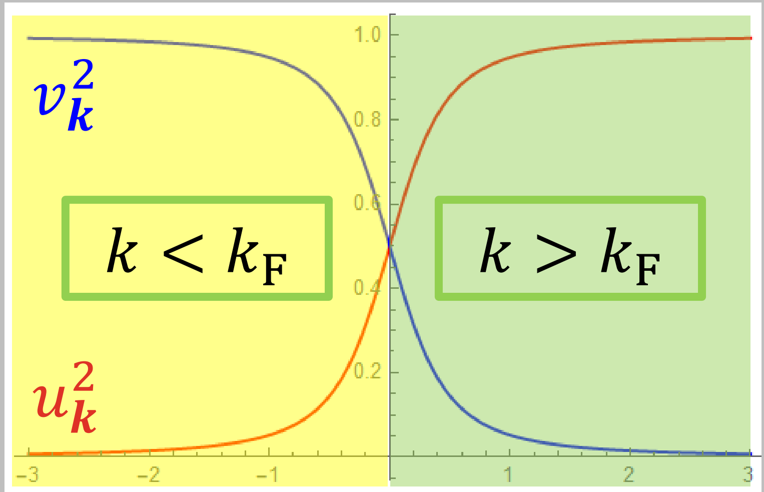
$$\Delta_{\boldsymbol{k}} = -\frac{1}{2}\sum_{\boldsymbol{q}} V(\boldsymbol{k}-\boldsymbol{q})\,\Delta_{\boldsymbol{q}} \Big/ \sqrt{\xi_{\boldsymbol{q}}^2 + \Delta_{\boldsymbol{q}}^2}$$

Zero temp. result!

Let us examine more about $|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}} \left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}} \hat{d}^\dagger_{\boldsymbol{k},\uparrow} \hat{d}^\dagger_{-\boldsymbol{k},\downarrow}\right)|0\rangle$





Route I: superconductivity in historical order

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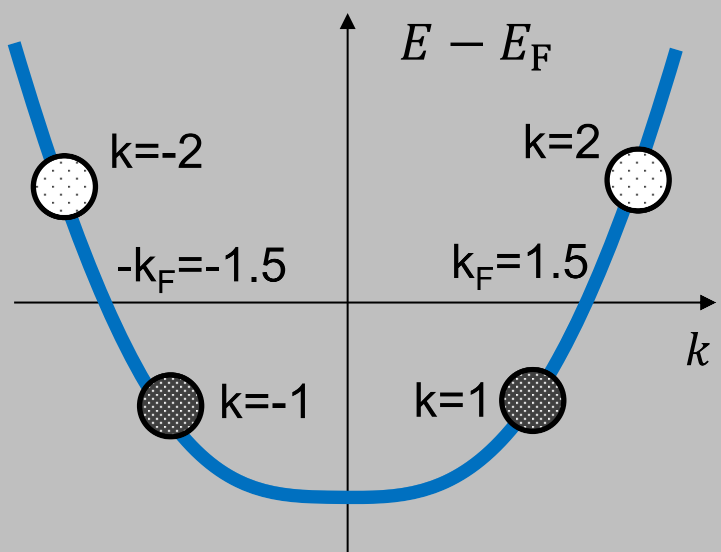
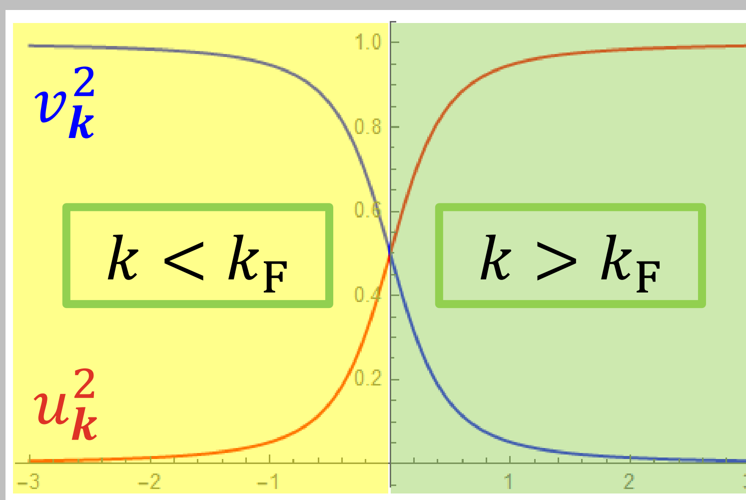
$$\Delta_{\boldsymbol{k}} = -\frac{1}{2}\sum_{\boldsymbol{q}} V(\boldsymbol{k}-\boldsymbol{q})\,\Delta_{\boldsymbol{q}} \Big/ \sqrt{\xi_{\boldsymbol{q}}^2 + \Delta_{\boldsymbol{q}}^2}$$

Zero temp. result!

Let us examine more about $|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}} \left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}} \hat{d}^{\dagger}_{\boldsymbol{k},\uparrow} \hat{d}^{\dagger}_{-\boldsymbol{k},\downarrow}\right)|0\rangle$



$$u^2_{\boldsymbol{k}=\pm 2} = \frac{3}{4}$$
$$v^2_{\boldsymbol{k}=\pm 2} = \frac{1}{4}$$
$$u^2_{\boldsymbol{k}=\pm 1} = \frac{1}{4}$$
$$v^2_{\boldsymbol{k}=\pm 1} = \frac{3}{4}$$



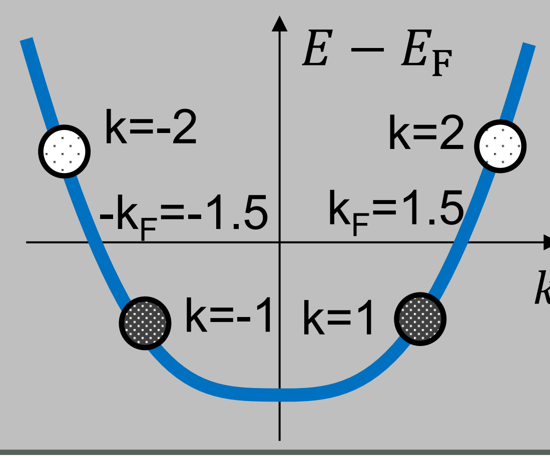
Route I: superconductivity in historical order

Let us examine more about $|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}}(u_{\boldsymbol{k}} + v_{\boldsymbol{k}} \hat{d}^{\dagger}_{\boldsymbol{k},\uparrow} \hat{d}^{\dagger}_{-\boldsymbol{k},\downarrow})|0\rangle$

$$|\mathrm{BCS}\rangle = \left(u_{-2} + v_{-2}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(u_{-1} + v_{-1}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right) \times \left(u_{1} + v_{1}\hat{d}^{\dagger}_{1,\uparrow}\hat{d}^{\dagger}_{-1,\downarrow}\right)\left(u_{2} + v_{2}\hat{d}^{\dagger}_{2,\uparrow}\hat{d}^{\dagger}_{-2,\downarrow}\right)|0\rangle$$

$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{1,\uparrow}\hat{d}^{\dagger}_{-1,\downarrow}\right) \times \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{2,\uparrow}\hat{d}^{\dagger}_{-2,\downarrow}\right)|0_{-2,\uparrow}, 0_{-2,\downarrow}, 0_{-1,\uparrow}, 0_{-1,\downarrow} 0_{1,\uparrow}, 0_{1,\downarrow} 0_{2,\uparrow}, 0_{2,\downarrow}\rangle$$

Typo.! Square roots for u & v's are omitted



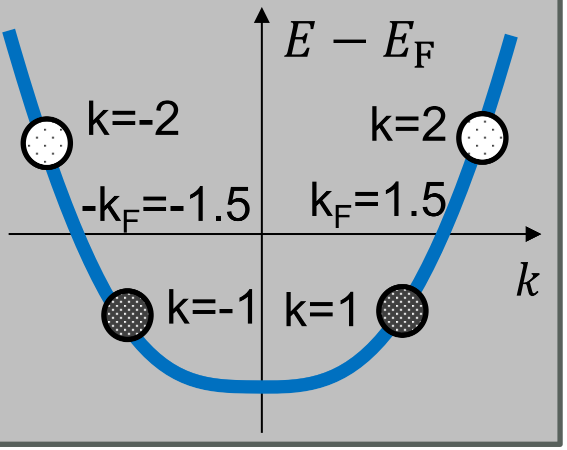
Route I: superconductivity in historical order

Let us examine more about $|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}}(u_{\boldsymbol{k}} + v_{\boldsymbol{k}}\hat{d}^{\dagger}_{\boldsymbol{k},\uparrow}\hat{d}^{\dagger}_{-\boldsymbol{k},\downarrow})|0\rangle$

$$|\mathrm{BCS}\rangle = (u_{-2} + v_{-2}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow})(u_{-1} + v_{-1}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}) \times (u_{1} + v_{1}\hat{d}^{\dagger}_{1,\uparrow}\hat{d}^{\dagger}_{-1,\downarrow})(u_{2} + v_{2}\hat{d}^{\dagger}_{2,\uparrow}\hat{d}^{\dagger}_{-2,\downarrow})|0\rangle$$

$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{1,\uparrow}\hat{d}^{\dagger}_{-1,\downarrow}\right) \times \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{2,\uparrow}\hat{d}^{\dagger}_{-2,\downarrow}\right)|0_{-2,\uparrow}, 0_{-2,\downarrow}, 0_{-1,\uparrow}, 0_{-1,\downarrow} 0_{1,\uparrow}, 0_{1,\downarrow} 0_{2,\uparrow}, 0_{2,\downarrow}\rangle$$

$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{1,\uparrow}\hat{d}^{\dagger}_{-1,\downarrow}\right) \times \left(\frac{3}{4}|0\rangle + \frac{1}{4}|1_{-2,\downarrow}, 1_{2,\uparrow}\rangle\right)$$



Route I: superconductivity in historical order

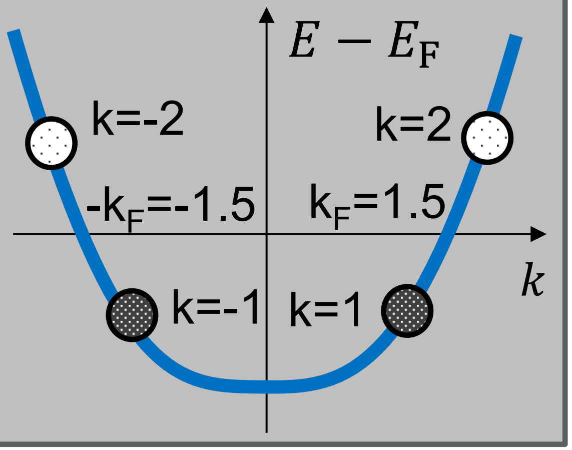
Let us examine more about $|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}}(u_{\boldsymbol{k}} + v_{\boldsymbol{k}}\hat{d}^{\dagger}_{\boldsymbol{k},\uparrow}\hat{d}^{\dagger}_{-\boldsymbol{k},\downarrow})|0\rangle$

$$|\mathrm{BCS}\rangle = (u_{-2} + v_{-2}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow})(u_{-1} + v_{-1}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow})$$

$$\times (u_1 + v_1\hat{d}^{\dagger}_{1,\uparrow}\hat{d}^{\dagger}_{-1,\downarrow})\,(u_2 + v_2\hat{d}^{\dagger}_{2,\uparrow}\hat{d}^{\dagger}_{-2,\downarrow})|0\rangle$$

$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{1,\uparrow}\hat{d}^{\dagger}_{-1,\downarrow}\right)$$

$$\times \left(\frac{3}{4}|0\rangle + \frac{1}{4}\left|1_{-2,\downarrow}, 1_{2,\uparrow}\right\rangle\right)$$



Route I:
superconductivity in historical order

Let us examine more about $|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}}\left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}}\hat{d}^{\dagger}_{\boldsymbol{k},\uparrow}\hat{d}^{\dagger}_{-\boldsymbol{k},\downarrow}\right)|0\rangle$

$$|\mathrm{BCS}\rangle = \left(u_{-2} + v_{-2}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(u_{-1} + v_{-1}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right) \times \left(u_{1} + v_{1}\hat{d}^{\dagger}_{1,\uparrow}\hat{d}^{\dagger}_{-1,\downarrow}\right)\left(u_{2} + v_{2}\hat{d}^{\dagger}_{2,\uparrow}\hat{d}^{\dagger}_{-2,\downarrow}\right)|0\rangle$$

$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{1,\uparrow}\hat{d}^{\dagger}_{-1,\downarrow}\right) \times \left(\frac{3}{4}|0\rangle + \frac{1}{4}\left|1_{-2,\downarrow}, 1_{2,\uparrow}\right\rangle\right)$$

$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right) \times \left(\frac{3}{4^2}|0\rangle + \frac{1}{4^2}\left|1_{-2,\downarrow}, 1_{2,\uparrow}\right\rangle + \frac{3^2}{4^2}\left|1_{-1,\downarrow}, 1_{1,\uparrow}\right\rangle + \frac{1}{4^2}\left|1_{-2,\downarrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{2,\uparrow}\right\rangle\right)$$

Route I:
superconductivity in historical order

Let us examine more about $|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}}(u_{\boldsymbol{k}} + v_{\boldsymbol{k}}\hat{d}^\dagger_{\boldsymbol{k},\uparrow}\hat{d}^\dagger_{-\boldsymbol{k},\downarrow})|0\rangle$

$$|\mathrm{BCS}\rangle = (u_{-2} + v_{-2}\hat{d}^\dagger_{-2,\uparrow}\hat{d}^\dagger_{2,\downarrow})(u_{-1} + v_{-1}\hat{d}^\dagger_{-1,\uparrow}\hat{d}^\dagger_{1,\downarrow}) \times (u_1 + v_1\hat{d}^\dagger_{1,\uparrow}\hat{d}^\dagger_{-1,\downarrow})(u_2 + v_2\hat{d}^\dagger_{2,\uparrow}\hat{d}^\dagger_{-2,\downarrow})|0\rangle$$

$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^\dagger_{-2,\uparrow}\hat{d}^\dagger_{2,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^\dagger_{-1,\uparrow}\hat{d}^\dagger_{1,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^\dagger_{1,\uparrow}\hat{d}^\dagger_{-1,\downarrow}\right) \times \left(\frac{3}{4}|0\rangle + \frac{1}{4}|1_{-2,\downarrow}, 1_{2,\uparrow}\rangle\right)$$

$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^\dagger_{-2,\uparrow}\hat{d}^\dagger_{2,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^\dagger_{-1,\uparrow}\hat{d}^\dagger_{1,\downarrow}\right) \times \left(\frac{3}{4^2}|0\rangle + \frac{1}{4^2}|1_{-2,\downarrow}, 1_{2,\uparrow}\rangle + \frac{3^2}{4^2}|1_{-1,\downarrow}, 1_{1,\uparrow}\rangle + \frac{1}{4^2}|1_{-2,\downarrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{2,\uparrow}\rangle\right)$$

0-electrons
=0-CP

2-electrons
=1-CP

4-electrons
=2-CP

Route I:
superconductivity in historical order

Let us examine more about $|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}}\left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}}\hat{d}^{\dagger}_{\boldsymbol{k},\uparrow}\hat{d}^{\dagger}_{-\boldsymbol{k},\downarrow}\right)|0\rangle$

$$|\mathrm{BCS}\rangle = \left(u_{-2} + v_{-2}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(u_{-1} + v_{-1}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right)$$

$$\times\left(u_1 + v_1\hat{d}^{\dagger}_{1,\uparrow}\hat{d}^{\dagger}_{-1,\downarrow}\right)\left(u_2 + v_2\hat{d}^{\dagger}_{2,\uparrow}\hat{d}^{\dagger}_{-2,\downarrow}\right)|0\rangle$$

$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right)$$

$$\times\left(\frac{3}{4^2}|0\rangle + \frac{1}{4^2}\left|1_{-2,\downarrow}, 1_{2,\uparrow}\right\rangle + \frac{3^2}{4^2}\left|1_{-1,\downarrow}, 1_{1,\uparrow}\right\rangle + \frac{1}{4^2}\left|1_{-2,\downarrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{2,\uparrow}\right\rangle\right)$$

Route I: superconductivity in historical order

Let us examine more about $|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}}\left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}}\hat{d}^{\dagger}_{\boldsymbol{k},\uparrow}\hat{d}^{\dagger}_{-\boldsymbol{k},\downarrow}\right)|0\rangle$

$$|\mathrm{BCS}\rangle = \left(u_{-2} + v_{-2}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(u_{-1} + v_{-1}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right)$$

$$\times\left(u_{1} + v_{1}\hat{d}^{\dagger}_{1,\uparrow}\hat{d}^{\dagger}_{-1,\downarrow}\right)\left(u_{2} + v_{2}\hat{d}^{\dagger}_{2,\uparrow}\hat{d}^{\dagger}_{-2,\downarrow}\right)|0\rangle$$

$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right)$$

$$\times\left(\frac{3}{4^2}|0\rangle + \frac{1}{4^2}\left|1_{-2,\downarrow}, 1_{2,\uparrow}\right\rangle + \frac{3^2}{4^2}\left|1_{-1,\downarrow}, 1_{1,\uparrow}\right\rangle + \frac{1}{4^2}\left|1_{-2,\downarrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{2,\uparrow}\right\rangle\right)$$

$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)$$

$$\times\left(\begin{array}{c}\frac{3}{4^3}|0\rangle + \frac{1}{4^3}\left|1_{-2,\downarrow}, 1_{2,\uparrow}\right\rangle + \frac{3^2}{4^3}\left|1_{-1,\downarrow}, 1_{1,\uparrow}\right\rangle + \frac{1}{4^3}\left|1_{-2,\downarrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{2,\uparrow}\right\rangle \\ + \frac{3^2}{4^3}\left|1_{-1,\uparrow}, 1_{1,\downarrow}\right\rangle + \frac{3}{4^3}\left|1_{-2,\downarrow}, 1_{-1,\uparrow}, 1_{1,\downarrow}, 1_{2,\uparrow}\right\rangle + \frac{3^3}{4^3}\left|1_{-1,\uparrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{1,\downarrow}\right\rangle \\ + \frac{3}{4^3}\left|1_{-2,\downarrow}, 1_{-1,\uparrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{1,\downarrow}, 1_{2,\uparrow}\right\rangle\end{array}\right)$$

Route I: superconductivity in historical order

Let us examine more about $|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}}(u_{\boldsymbol{k}} + v_{\boldsymbol{k}}\hat{d}^{\dagger}_{\boldsymbol{k},\uparrow}\hat{d}^{\dagger}_{-\boldsymbol{k},\downarrow})|0\rangle$

$$|\mathrm{BCS}\rangle = (u_{-2} + v_{-2}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow})(u_{-1} + v_{-1}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow})$$
$$\times (u_1 + v_1\hat{d}^{\dagger}_{1,\uparrow}\hat{d}^{\dagger}_{-1,\downarrow})(u_2 + v_2\hat{d}^{\dagger}_{2,\uparrow}\hat{d}^{\dagger}_{-2,\downarrow})|0\rangle$$
$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)\left(\frac{1}{4} + \frac{3}{4}\hat{d}^{\dagger}_{-1,\uparrow}\hat{d}^{\dagger}_{1,\downarrow}\right)$$
$$\times \left(\frac{3}{4^2}|0\rangle + \frac{1}{4^2}|1_{-2,\downarrow}, 1_{2,\uparrow}\rangle + \frac{3^2}{4^2}|1_{-1,\downarrow}, 1_{1,\uparrow}\rangle + \frac{1}{4^2}|1_{-2,\downarrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{2,\uparrow}\rangle\right)$$
$$= \left(\frac{3}{4} + \frac{1}{4}\hat{d}^{\dagger}_{-2,\uparrow}\hat{d}^{\dagger}_{2,\downarrow}\right)$$
$$\times \left(\frac{3}{4^3}|0\rangle + \frac{1}{4^3}|1_{-2,\downarrow}, 1_{2,\uparrow}\rangle + \frac{3^2}{4^3}|1_{-1,\downarrow}, 1_{1,\uparrow}\rangle + \frac{1}{4^3}|1_{-2,\downarrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{2,\uparrow}\rangle \right.$$
$$+ \frac{3^2}{4^3}|1_{-1,\uparrow}, 1_{1,\downarrow}\rangle + \frac{3}{4^3}|1_{-2,\downarrow}, 1_{-1,\uparrow}, 1_{1,\downarrow}, 1_{2,\uparrow}\rangle + \frac{3^3}{4^3}|1_{-1,\uparrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{1,\downarrow}\rangle$$
$$\left. + \frac{3}{4^3}|1_{-2,\downarrow}, 1_{-1,\uparrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{1,\downarrow}, 1_{2,\uparrow}\rangle \right)$$

1-CP

2-CP

3-CP

Route I: superconductivity in historical order

Let us examine more about $|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}} \left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}} \hat{d}^{\dagger}_{\boldsymbol{k},\uparrow} \hat{d}^{\dagger}_{-\boldsymbol{k},\downarrow} \right) |0\rangle$

$$|\mathrm{BCS}\rangle = \left(\frac{3}{4} + \frac{1}{4} \hat{d}^{\dagger}_{-2,\uparrow} \hat{d}^{\dagger}_{2,\downarrow} \right)$$

$$\times \left(\frac{3}{4^3}|0\rangle + \frac{1}{4^3}|1_{-2,\downarrow}, 1_{2,\uparrow}\rangle + \frac{3^2}{4^3}|1_{-1,\downarrow}, 1_{1,\uparrow}\rangle + \frac{1}{4^3}|1_{-2,\downarrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{2,\uparrow}\rangle + \frac{3^2}{4^3}|1_{-1,\uparrow}, 1_{1,\downarrow}\rangle + \frac{3}{4^3}|1_{-2,\downarrow}, 1_{-1,\uparrow}, 1_{1,\downarrow}, 1_{2,\uparrow}\rangle + \frac{3^3}{4^3}|1_{-1,\uparrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{1,\downarrow}\rangle + \frac{3}{4^3}|1_{-2,\downarrow}, 1_{-1,\uparrow}, 1_{-1,\downarrow}, 1_{1,\uparrow}, 1_{1,\downarrow}, 1_{2,\uparrow}\rangle \right)$$

Route I: superconductivity in historical order

$$|\mathrm{BCS}\rangle = \frac{3^2}{4^4}|0\rangle + \frac{3}{4^4}|1_{-2,\downarrow},1_{2,\uparrow}\rangle + \frac{3^3}{4^4}|1_{-1,\downarrow},1_{1,\uparrow}\rangle + \frac{3}{4^4}|1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow}\rangle$$

$$+ \frac{3}{4^4}|1_{-1,\uparrow},1_{1,\downarrow}\rangle + \frac{3^2}{4^4}|1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle + \frac{3^4}{4^4}|1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow}\rangle$$

$$+ \frac{3^2}{4^4}|1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle$$

$$+ \frac{3}{4^4}|1_{-2,\uparrow},1_{2,\downarrow}\rangle + \frac{1}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle + \frac{3^2}{4^4}|1_{-2,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\downarrow}\rangle$$

$$+ \frac{1}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow},1_{2,\downarrow}\rangle + \frac{3^2}{4^4}|1_{-2,\uparrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle$$

$$+ \frac{3}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle + \frac{3^3}{4^4}|1_{-2,\uparrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle$$

$$+ \frac{3}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle$$

Route I:
superconductivity in historical order

$$|\mathrm{BCS}\rangle = \frac{3^2}{4^4}|0\rangle$$
$$+\frac{3}{4^4}|1_{-1,\uparrow},1_{1,\downarrow}\rangle+\frac{3^3}{4^4}|1_{-1,\downarrow},1_{1,\uparrow}\rangle+\frac{3}{4^4}|1_{-2,\uparrow},1_{2,\downarrow}\rangle+\frac{3}{4^4}|1_{-2,\downarrow},1_{2,\uparrow}\rangle$$

$$+\frac{3}{4^4}|1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow}\rangle+\frac{3^2}{4^4}|1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle+\frac{3^4}{4^4}|1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow}\rangle$$
$$+\frac{1}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle+\frac{3^2}{4^4}|1_{-2,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\downarrow}\rangle+\frac{3^2}{4^4}|1_{-2,\uparrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle$$

$$+\frac{3^2}{4^4}|1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle+\frac{1}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow},1_{2,\downarrow}\rangle$$
$$+\frac{3}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle+\frac{3^3}{4^4}|1_{-2,\uparrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle$$

$$+\frac{3}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle$$

$$u_{\boldsymbol{k}=\pm2}=\frac{3}{4}\ \&\ v_{\boldsymbol{k}=\pm2}=\frac{1}{4}$$
$$u_{\boldsymbol{k}=\pm1}=\frac{1}{4}\ \&\ v_{\boldsymbol{k}=\pm1}=\frac{3}{4}$$

=

$$u_{\boldsymbol{k}}=\begin{cases}1/4\ \ (k<k_{\mathrm{F}})\\ 3/4\,(k>k_{\mathrm{F}})\end{cases}$$
$$v_{\boldsymbol{k}}=\begin{cases}1/4\,(k<k_{\mathrm{F}})\\ 3/4\,(k>k_{\mathrm{F}})\end{cases}$$

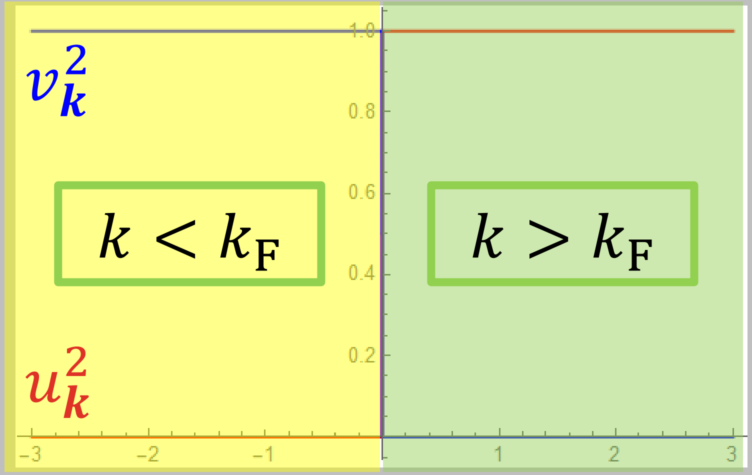
Route I: superconductivity in historical order

$$|\mathrm{BCS}\rangle = \frac{3^2}{4^4}|0\rangle$$
$$+\frac{3}{4^4}|1_{-1,\uparrow},1_{1,\downarrow}\rangle+\frac{3^3}{4^4}|1_{-1,\downarrow},1_{1,\uparrow}\rangle+\frac{3}{4^4}|1_{-2,\uparrow},1_{2,\downarrow}\rangle+\frac{3}{4^4}|1_{-2,\downarrow},1_{2,\uparrow}\rangle$$
$$+\frac{3}{4^4}|1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow}\rangle+\frac{3^2}{4^4}|1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle+\frac{3^4}{4^4}|1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow}\rangle$$
$$+\frac{1}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle+\frac{3^2}{4^4}|1_{-2,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\downarrow}\rangle+\frac{3^2}{4^4}|1_{-2,\uparrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle$$
$$+\frac{3^2}{4^4}|1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle+\frac{1}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow},1_{2,\downarrow}\rangle$$
$$+\frac{3}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle+\frac{3^3}{4^4}|1_{-2,\uparrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle$$
$$+\frac{3}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle$$

What happens $\Delta \to 0$ limits?

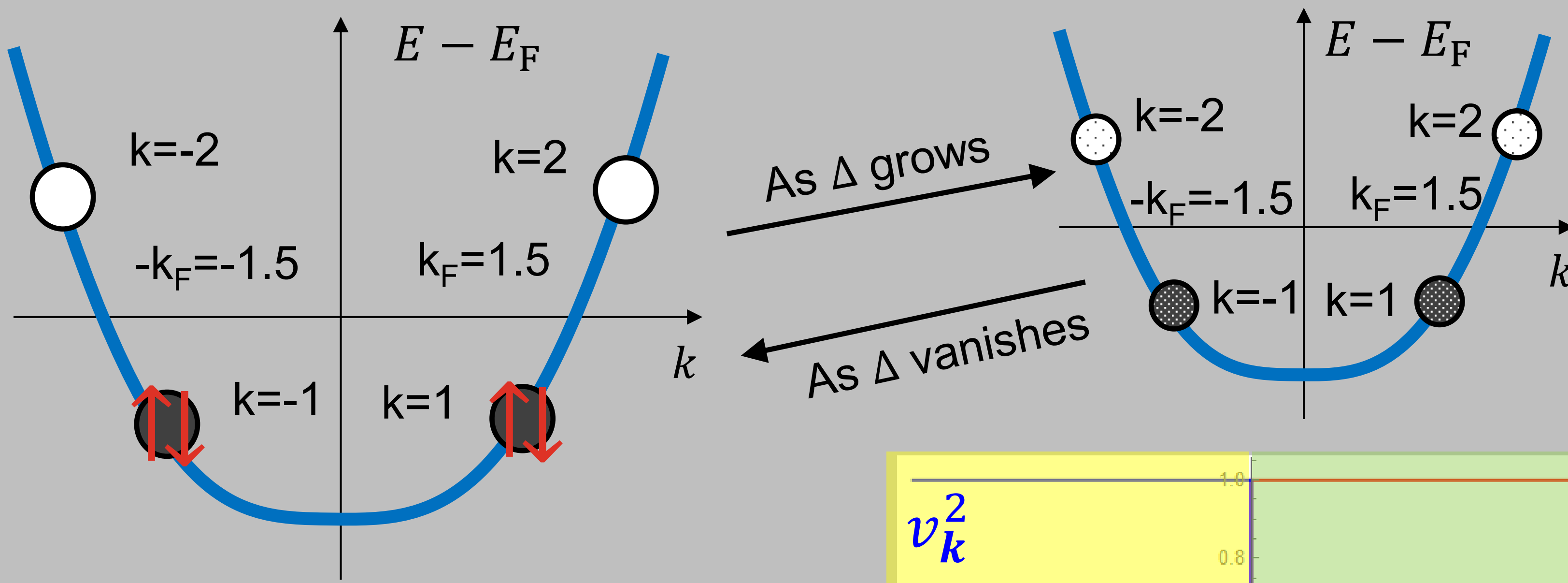
$$u_k = \begin{cases} 1/4 & (k < k_F) \\ 3/4 & (k > k_F) \end{cases} \quad v_k = \begin{cases} 1/4 & (k < k_F) \\ 3/4 & (k > k_F) \end{cases}$$

$$u_k = \begin{cases} 0 & (k < k_F) \\ 1 & (k > k_F) \end{cases} \quad v_k = \begin{cases} 1 & (k < k_F) \\ 0 & (k > k_F) \end{cases}$$



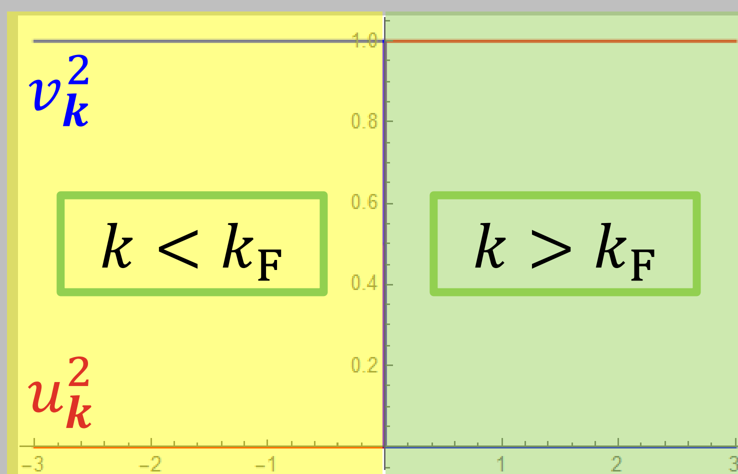
Route I:
superconductivity in historical order

$$\lim_{\Delta\to 0}|\mathrm{BCS}\rangle = |1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow}\rangle = |\mathrm{FS}\rangle$$



$$u_k^2 = \begin{cases} 1/4 & (k<k_\mathrm{F}) \\ 3/4 & (k>k_\mathrm{F}) \end{cases}$$
$$v_k^2 = \begin{cases} 1/4 & (k<k_\mathrm{F}) \\ 3/4 & (k>k_\mathrm{F}) \end{cases}$$

$$u_k = \begin{cases} 0 & (k<k_\mathrm{F}) \\ 1 & (k>k_\mathrm{F}) \end{cases}$$
$$v_k = \begin{cases} 1 & (k<k_\mathrm{F}) \\ 0 & (k>k_\mathrm{F}) \end{cases}$$



Route I:
superconductivity in historical order

$$|\mathrm{BCS}\rangle = \frac{3^2}{4^4}|0\rangle$$
$$+\frac{3}{4^4}|1_{-1,\uparrow},1_{1,\downarrow}\rangle+\frac{3^3}{4^4}|1_{-1,\downarrow},1_{1,\uparrow}\rangle+\frac{3}{4^4}|1_{-2,\uparrow},1_{2,\downarrow}\rangle+\frac{3}{4^4}|1_{-2,\downarrow},1_{2,\uparrow}\rangle$$
$$+\frac{3}{4^4}|1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow}\rangle+\frac{3^2}{4^4}|1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle+\frac{3^4}{4^4}|1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow}\rangle$$
$$+\frac{1}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle+\frac{3^2}{4^4}|1_{-2,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\downarrow}\rangle+\frac{3^2}{4^4}|1_{-2,\uparrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle$$
$$+\frac{3^2}{4^4}|1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle+\frac{1}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow},1_{2,\downarrow}\rangle$$
$$+\frac{3}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle+\frac{3^3}{4^4}|1_{-2,\uparrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle$$
$$+\frac{3}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle$$

What happens $\Delta \to \infty$ limits?

→ Particle number is uncertain and $[\hat{\Phi},\hat{N}]\sim 1$

→ SC phase $\boldsymbol{\Phi}$ is a good dynamical variable

Macroscopic wavefunction for SCs

$$|\boldsymbol{\Phi}\rangle = |0\rangle + e^{i\boldsymbol{\Phi}}|2e\rangle + e^{2i\boldsymbol{\Phi}}|4e\rangle + \cdots$$

(unbreakable Cooper pairs, i.e., **all relevant energy scales << Δ**)

Route I: superconductivity in historical order

$$|\mathrm{BCS}\rangle = \frac{3^2}{4^4}|0\rangle$$
$$+\frac{3}{4^4}|1_{-1,\uparrow},1_{1,\downarrow}\rangle+\frac{3^3}{4^4}|1_{-1,\downarrow},1_{1,\uparrow}\rangle+\frac{3}{4^4}|1_{-2,\uparrow},1_{2,\downarrow}\rangle+\frac{3}{4^4}|1_{-2,\downarrow},1_{2,\uparrow}\rangle$$
$$+\frac{3}{4^4}|1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow}\rangle+\frac{3^2}{4^4}|1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle+\frac{3^4}{4^4}|1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow}\rangle$$
$$+\frac{1}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle+\frac{3^2}{4^4}|1_{-2,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\downarrow}\rangle+\frac{3^2}{4^4}|1_{-2,\uparrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle$$
$$+\frac{3^2}{4^4}|1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle+\frac{1}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow},1_{2,\downarrow}\rangle$$
$$+\frac{3}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle+\frac{3^3}{4^4}|1_{-2,\uparrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle$$
$$+\frac{3}{4^4}|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle$$

What happens $\Delta \to \infty$ limits?

$$u_{\boldsymbol{k}}^2 = \begin{cases} 1/4 & (k<k_\mathrm{F}) \\ 3/4 & (k>k_\mathrm{F}) \end{cases}$$
$$v_{\boldsymbol{k}}^2 = \begin{cases} 1/4 & (k<k_\mathrm{F}) \\ 3/4 & (k>k_\mathrm{F}) \end{cases}$$

$$u_{\boldsymbol{k}}^2 = \frac{1}{2}\left(1+\frac{\xi_{\boldsymbol{k}}}{\sqrt{\xi_{\boldsymbol{k}}^2+\Delta_{\boldsymbol{k}}^2}}\right) \to \frac{1}{2}$$
$$v_{\boldsymbol{k}}^2 = \frac{1}{2}\left(1-\frac{\xi_{\boldsymbol{k}}}{\sqrt{\xi_{\boldsymbol{k}}^2+\Delta_{\boldsymbol{k}}^2}}\right) \to \frac{1}{2}$$

Route I: superconductivity in historical order

$$\lim_{\Delta\to\infty} |\mathrm{BCS}\rangle = \left(\frac{1}{\sqrt{2}}\right)^4 [|0\rangle$$

0-Cooper pair

$$+|1_{-1,\uparrow},1_{1,\downarrow}\rangle + |1_{-1,\downarrow},1_{1,\uparrow}\rangle + |1_{-2,\uparrow},1_{2,\downarrow}\rangle + |1_{-2,\downarrow},1_{2,\uparrow}\rangle$$

1-Cooper pair

2-Cooper pairs

$$+|1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow}\rangle + |1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle + |1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow}\rangle$$
$$+|1_{-2,\uparrow},1_{-2,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle + |1_{-2,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\downarrow}\rangle + |1_{-2,\uparrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle$$

$$+|1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle + |1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow},1_{2,\downarrow}\rangle$$
$$+|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle + |1_{-2,\uparrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle$$

3-Cooper pairs

$$+|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle]$$

4-Cooper pairs

Check the normalization

$$\left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^4 + 6\left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = \frac{(1+1)^4}{2^4} = 1$$

0-CP 1-CP 2-CP 3-CP 4-CP

Route I:
superconductivity in historical order

$$\lim_{\Delta\to\infty} |\mathrm{BCS}\rangle = \left(\frac{1}{\sqrt{2}}\right)^4 [|0\rangle$$

$|0e\rangle$

$|2e\rangle$

$$+\left(|1_{-1,\uparrow},1_{1,\downarrow}\rangle + |1_{-1,\downarrow},1_{1,\uparrow}\rangle + |1_{-2,\uparrow},1_{2,\downarrow}\rangle + |1_{-2,\downarrow},1_{2,\uparrow}\rangle\right)\left(e^{i\varphi/2}\right)^2$$

$|4e\rangle$

$$+\left(|1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow}\rangle + |1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle + |1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow}\rangle + |1_{-2,\uparrow},1_{-2,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle + |1_{-2,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\downarrow}\rangle + |1_{-2,\uparrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle\right)\left(e^{i\varphi/2}\right)^4$$

$$+\left(|1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow}\rangle + |1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\downarrow},1_{1,\uparrow},1_{2,\uparrow},1_{2,\downarrow}\rangle + |1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle + |1_{-2,\uparrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\downarrow}\rangle\right)\left(e^{i\varphi/2}\right)^6$$

$|6e\rangle$

$$+|1_{-2,\uparrow},1_{-2,\downarrow},1_{-1,\uparrow},1_{-1,\downarrow},1_{1,\uparrow},1_{1,\downarrow},1_{2,\uparrow},1_{2,\downarrow}\rangle\left(e^{i\varphi/2}\right)^8]$$

$|8e\rangle$

Previously, we did

$$|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}}\left(|u_{\boldsymbol{k}}| + |v_{\boldsymbol{k}}|e^{i\varphi}\hat{d}^\dagger_{\boldsymbol{k},\uparrow}\hat{d}^\dagger_{-\boldsymbol{k},\downarrow}\right)|0\rangle$$
$$= \prod_{\boldsymbol{k}}\left(|u_{\boldsymbol{k}}| + |v_{\boldsymbol{k}}|\hat{d}^\dagger_{\boldsymbol{k},\uparrow}\hat{d}^\dagger_{-\boldsymbol{k},\downarrow}\right)|0\rangle$$

(by doing $e^{i\varphi/2}\hat{d}^\dagger_{\boldsymbol{k},\sigma} \mapsto \hat{d}^\dagger_{\boldsymbol{k},\sigma}$)

Doing reverse $\hat{d}^\dagger_{\boldsymbol{k},\sigma} \mapsto e^{i\varphi/2}\hat{d}^\dagger_{\boldsymbol{k},\sigma}$

Route I: superconductivity in historical order

Macroscopic wavefunction for SCs

$$|\varphi\rangle = |0\rangle + e^{i\varphi}|2e\rangle + e^{2i\varphi}|4e\rangle + \cdots$$

(unbreakable Cooper pairs, i.e., **all relevant energy scales << Δ**)

$$\lim_{\Delta\to\infty} |\mathrm{BCS}\rangle = \sum_{n=0}^{N=4} e^{in\varphi}|2ne\rangle = |\varphi\rangle$$

→ Particle number can be uncertain and $[\hat{\varphi},\hat{n}]\sim 1$

→ For $|\varphi\rangle$, SC phase $\hat{\varphi}$ is a good dynamical variable

→ For $|\varphi\rangle$, SC phase $\hat{n}$ is infinitely uncertain

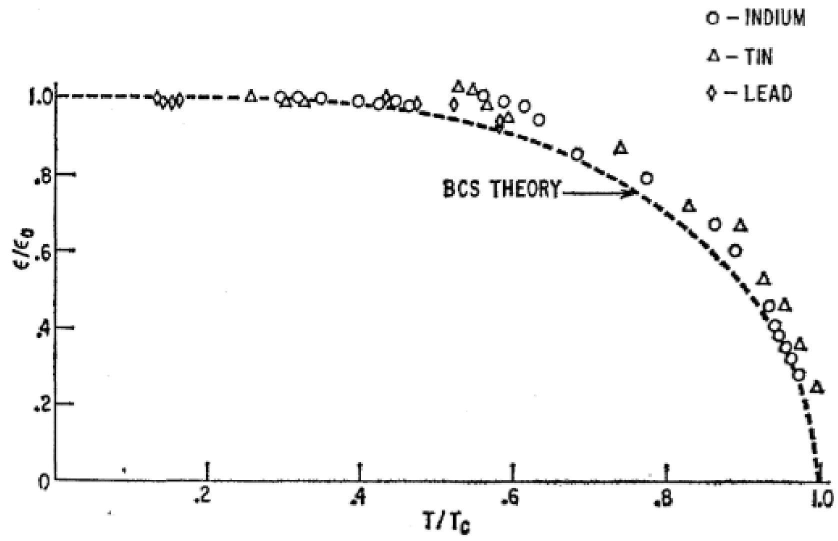
Additional note

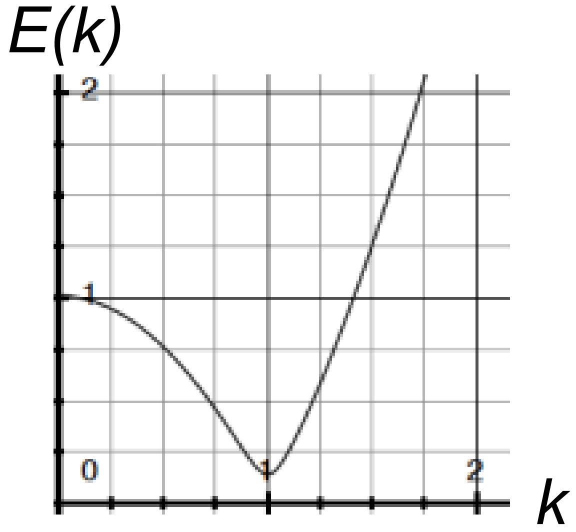
→ $|\varphi\rangle$ is for bulk superconductors (no energy cost for many electrons)

→ When SC is small & charge energy matters, another energy minimizing superposition of Cooper pairs is the ground state of the small superconductor

Route I:
superconductivity in historical order

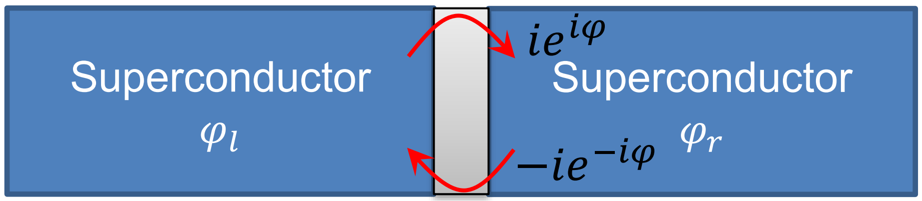
- **What BCS theory provides**
  → Excitation spectrum (Bogoliubov quasiparticle)
  → SC gap as a function of temperature & $2\Delta(0) \approx 3.53 k_B T_c$
  → Partition function as a function of temperature,
  predicting heat capacity, $\Delta C = 1.43\gamma T_c$ & $C_v \propto e^{-\Delta/k_B T}$
  → Isotope effect: $T_c \propto M^{-1/2}$
  → Meissner effect & $H_c^2 = 4\pi N(0)\Delta^2$
  → and so many!





Route I: superconductivity in historical order

- **Josephson junction & its descriptions**



→ In superconductor with vanishing phase uncertainty, charge can fluctuate indefinitely, **N-1 = N & N+1 = N**.

$$H_J \propto \sum_{N_r,N_l} (|N_l - 1\rangle\langle N_l| \otimes |N_r + 1\rangle\langle N_r| + h.c.)$$

→ $|\mathrm{GS}\rangle = |\varphi_l\rangle \otimes |\varphi_r\rangle$ before making contact
= $|\mathrm{GS}\rangle$ after making the contact, where $|\varphi\rangle = \sum_{n=0}^{\infty} e^{in\varphi}|n\rangle$

→ Josephson energy = $\langle \mathrm{GS}|H_J|\mathrm{GS}\rangle \propto \cos(\varphi_r - \varphi_l)$

$$\left(\sum_N |N-1\rangle\langle N|\right)|\varphi\rangle = \sum_{n=0}^{\infty} e^{in\varphi}|n-1\rangle = e^{i\varphi}\sum_{n=0}^{\infty} e^{i(n-1)\varphi}|n-1\rangle = e^{i\varphi}|\varphi\rangle$$

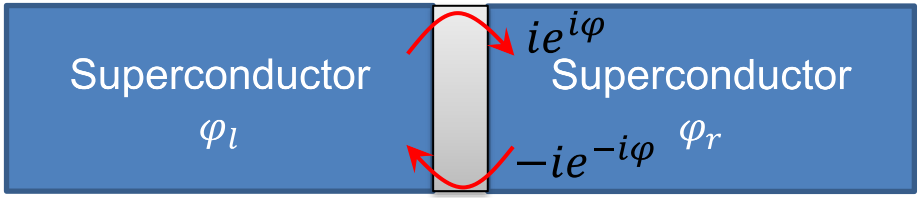
→ Current operator, $\hat{J} \propto i\sum_{N_r,N_l}(|N_l - 1\rangle\langle N_l| \otimes |N_r + 1\rangle\langle N_r| - h.c)$

Pair current = $\langle \mathrm{GS}|\hat{J}|\mathrm{GS}\rangle \propto \sin(\varphi_r - \varphi_l)$

Applicable for finite Δ if Δ is the largest energy scale, e.g., slow & small electrical bias

Route I: superconductivity in historical order

- **Josephson junction & its descriptions**



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→ Current operator, $\hat{J} \propto i\sum_{N_r,N_l}(|N_l - 1\rangle\langle N_l| \otimes |N_r + 1\rangle\langle N_r| - h.c)$

Pair current $=\langle \mathrm{GS}|\hat{J}|\mathrm{GS}\rangle \propto \sin(\varphi_r - \varphi_l)$

Applicable for finite Δ if Δ is the largest energy scale, e.g., slow & small electrical bias

Route I: superconductivity in historical order

- **Josephson effect**

  → In superconductor with vanishing phase uncertainty, charge can fluctuate indefinitely, N-1 = N & N+1 = N.

$$H_J \propto \sum_{N_r,N_l} (|N_r+1\rangle\langle N_l-1| + h.c.)$$

→ $|\mathrm{GS}\rangle = |\varphi_l\rangle \otimes |\varphi_r\rangle$ with the contact

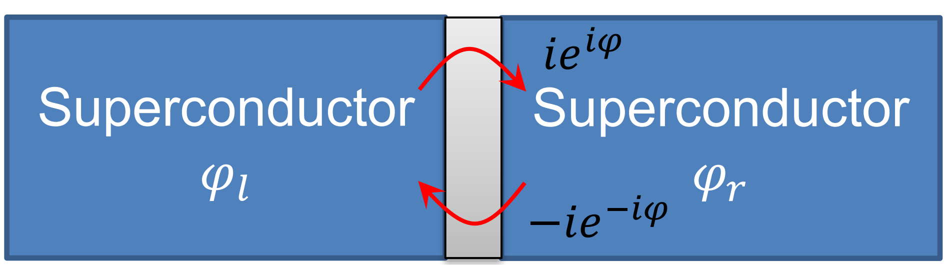
→ Josephson energy = $\langle \mathrm{GS}|H_J|\mathrm{GS}\rangle \propto \cos(\varphi_r - \varphi_l)$

→ Pair current = $\langle \mathrm{GS}|\hat{J}|\mathrm{GS}\rangle \propto \sin(\varphi_r - \varphi_l)$

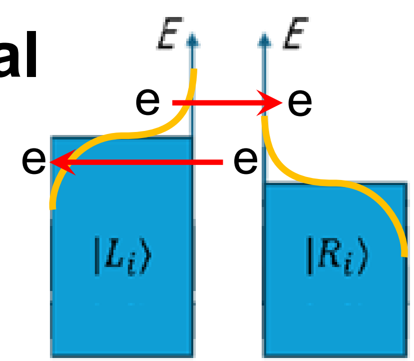
→ In SC, two process can interfere, differently from normal metals

SC = pure ensemble V.S. Normal metal = mixed ensemble

$P = |A+B|^2$



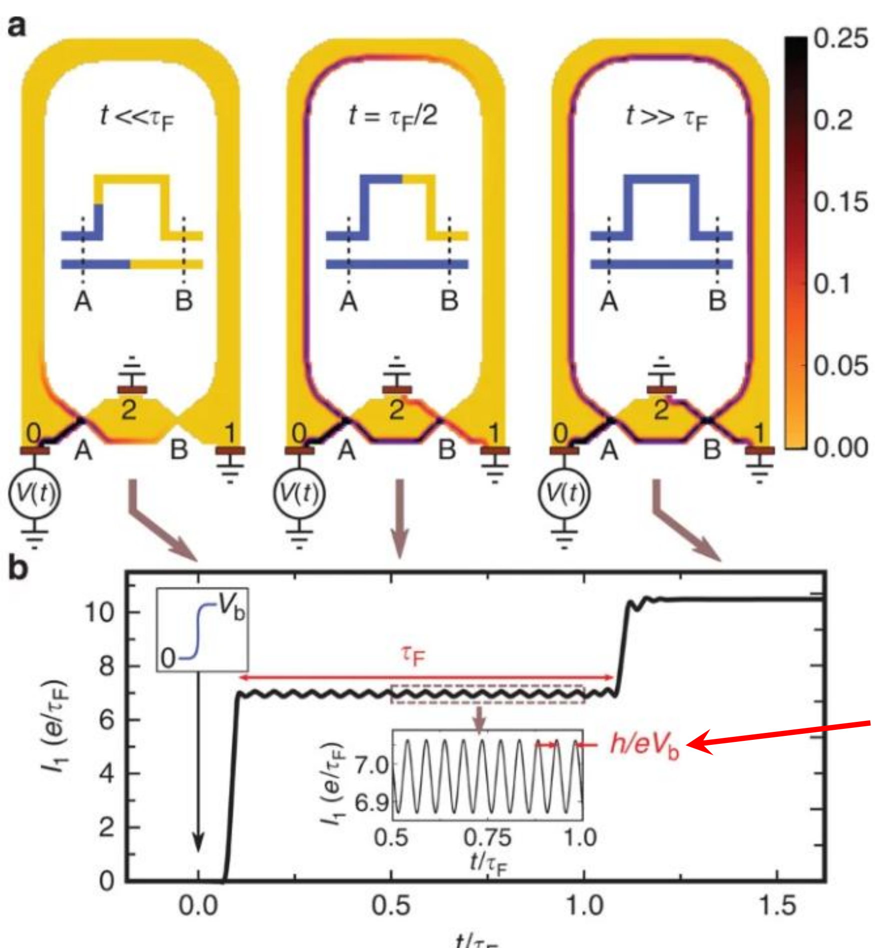
$P = |A|^2 + |B|^2$

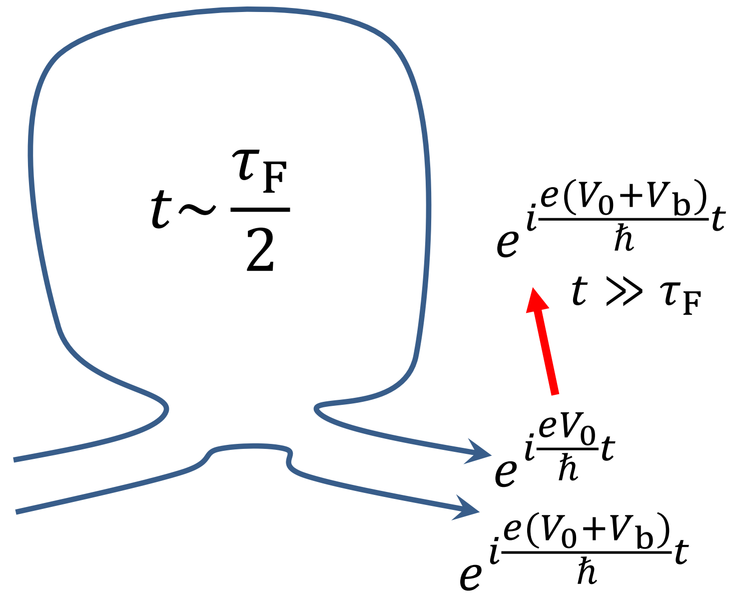


$|i\rangle = |L_i\rangle \otimes |R_i\rangle$

Route I:
superconductivity in historical order

- **a.c. Josephson effect without superconductivity**
  → Quantum interference in Mach-Zehnder interferometry





Half the superconducting Josephson frequency, $f = e^* V_b / h$

B. Gaury, J. Weston, & X. Waintal, *The a.c. Josephson effect without superconductivity*, Nat. Commun. **6**, 6524 (2015).

Route I:
superconductivity in historical order

- **The 1st and 2nd Josephson relations Josephson effect**

$$I_s = I_c \sin\varphi$$ (weak-link current-phase relation)

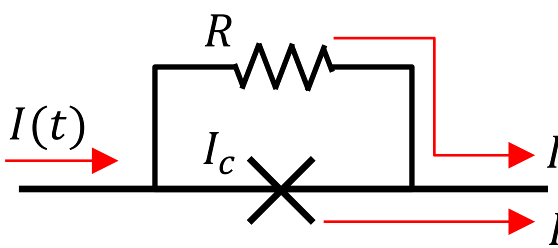
→ Physical origin: quantum interference of Cooper pair tunneling

$$\varphi'(t) = e^* V(t)/\hbar$$ (SC phase evolution)

→ Physical origin: energy difference between superconductors

- **Widely used theoretical models**

→ **Resistively and Capacitively Shunted Junction (RCSJ model)**



$$I_n(t) = \frac{v(t)}{R}$$

$$I_s(t) = I_c \sin\varphi(t)$$

Phenomenological circuit for Josephson junctions

With AC Josephson effect $v(t) = \frac{\hbar}{2e}\frac{d\varphi}{dt}$

$$I(t) = I_n(t) + I_s(t)$$
$$= \frac{\hbar}{2eR_n}\frac{d\varphi}{dt} + I_c \sin\varphi(t)$$

W. C. Stewart, Current-Voltage Characteristics of Josephson junctions, Appl. Phys. Lett. **12**, 277 (1968).

Route I:
superconductivity in historical order

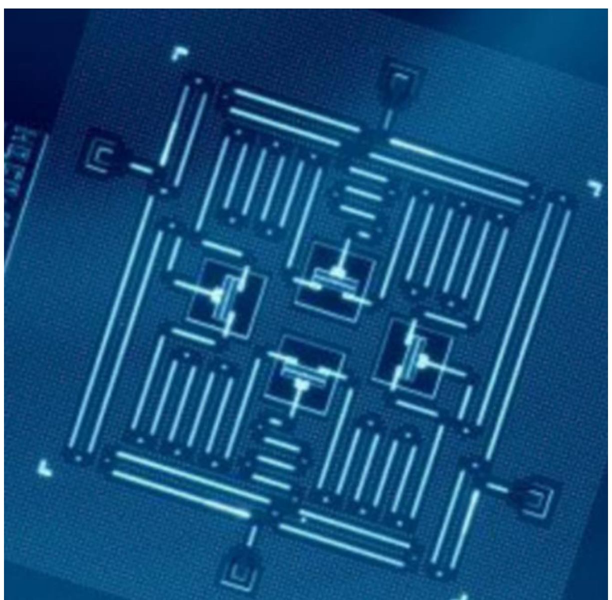
- **Widely used theoretical models**
  → Number-phase representation of BCS state

$$|\varphi\rangle = \sum_{n=0}^{\infty} e^{in\varphi}|n\rangle$$

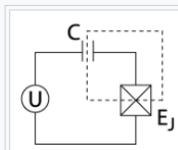
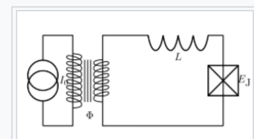
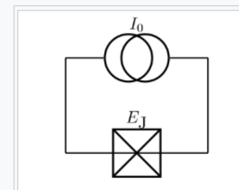
$$|n\rangle \equiv \int_0^{2\pi} d\varphi\, e^{-in\varphi/2}|\mathrm{BCS}\rangle$$

Detailed discussion: P. W. Anderson, Rev. Mod. Phys. **38**, 298 (1966).
Textbook introduction: M. Tinkham, *Introduction to Superconductivity*, McGraw-Hill, 2nd ed., p.256 (1996).

→ Frequenctly used to describe qubits

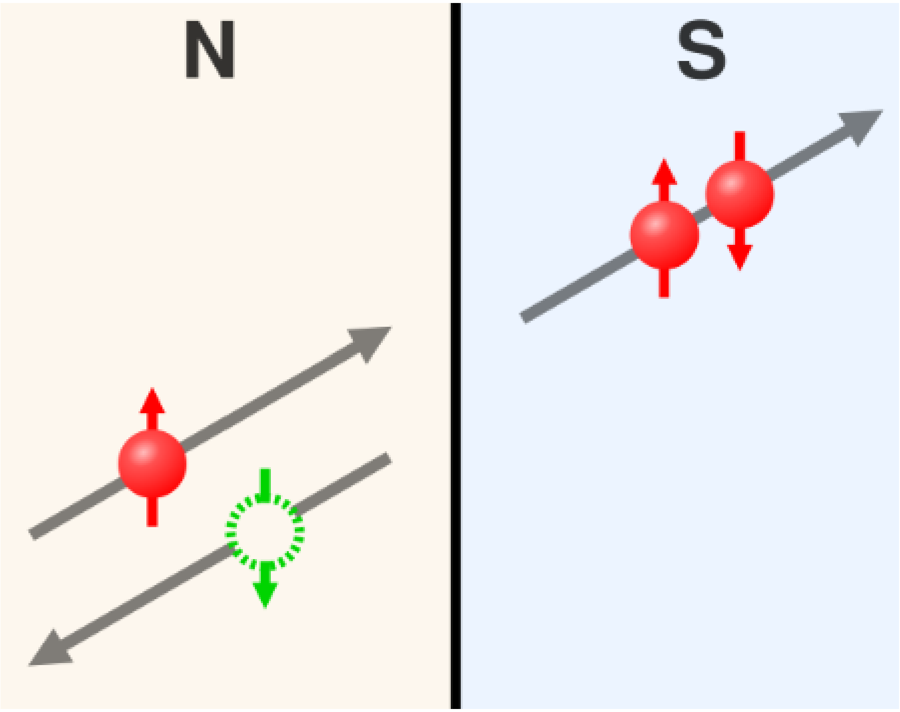


Superconducting Qubit Archetypes [38]

Aspect / Type	Charge qubit	RF-SQUID qubit (prototype of the Flux Qubit)	Phase qubit
Circuit	 Charge qubit circuit. A superconducting island (encircled with a dashed line) is defined between the leads of a capacitor with capacitance C and a Josephson junction with energy E_J biased by voltage U.	 Flux qubit circuit. A superconducting loop with inductance L is interrupted by a junction with Josephson energy E_J. Bias flux Φ is induced by a flux line with current I_0.	 Phase qubit circuit. A Josephson junction with energy parameter E_J is biased by current I_0.
Hamiltonian	$H = E_C(N - N_g)^2 - E_J \cos\phi$ In this case N is the number of Cooper pairs to tunnel through the junction, $N_g = CV_0/2e$ is the charge on the capacitor in units of Cooper pairs number,	$H = \frac{q^2}{2C_J} + \left(\frac{\Phi_0}{2\pi}\right)^2 \frac{\phi^2}{2L} - E_J \cos\left[\phi - \Phi\frac{2\pi}{\Phi_0}\right]$ Note that ϕ is only allowed to take values greater	$H = \frac{(2e)^2}{2C_J}q^2 - I_0\frac{\Phi_0}{2\pi}\phi - E_J\cos\phi$

Route I: superconductivity in historical order

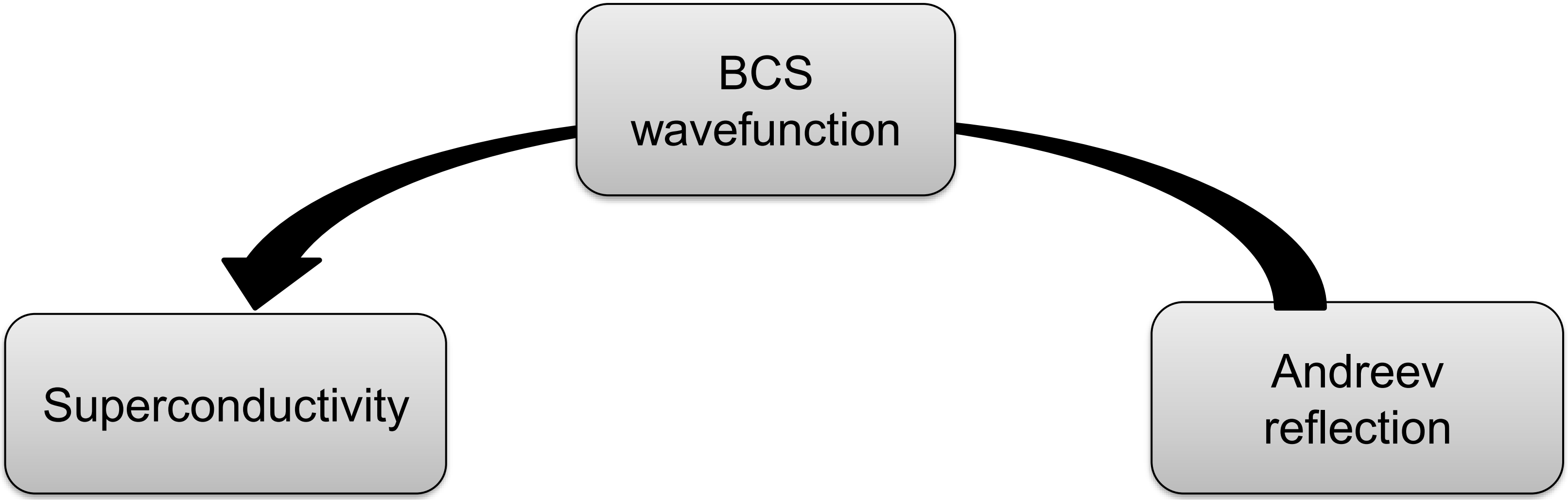
- **Andreev reflection**
  → particle-hole conversion



A. F. Andreev, *The Thermal Conductivity of the Intermediate State of Superconductors*, Soviet Physics JETP **19**, 1228 (1964)

7 yrs later after BCS theory

Route II:
mesoscopic superconductors as a black box providing Andreev reflection (AR)

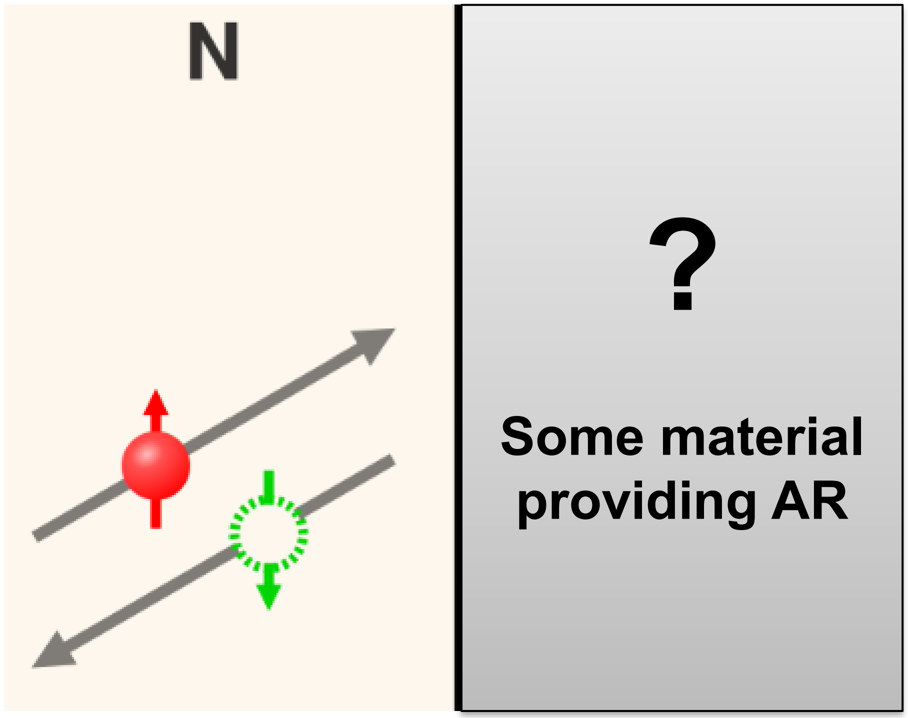


Route II:
mesoscopic superconductors from AR

- **Our starting point: Andreev reflection (AR)**

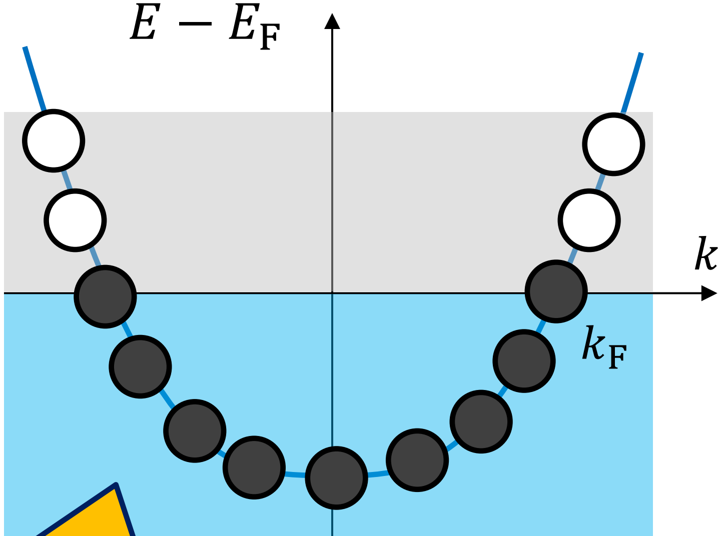
'superconductivity is something providing AR'

& we don't have to be involved with many-body physics



Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**



In momentum space

Ground state = Fermi sea (FS)

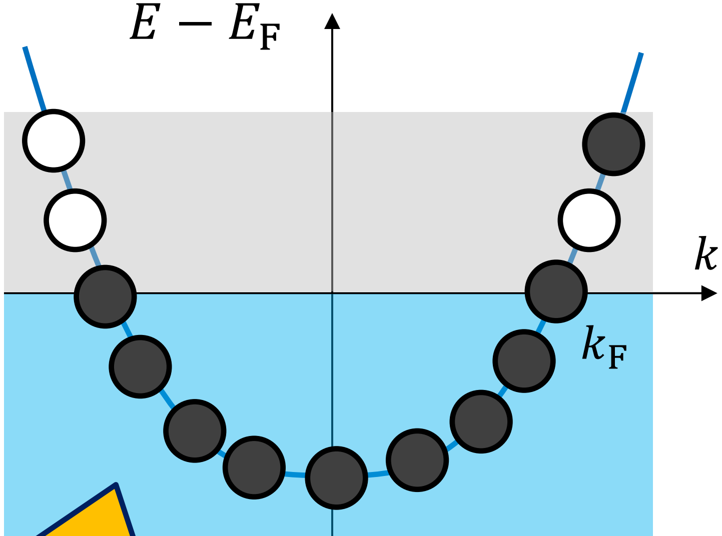
$$|\mathrm{FS}\rangle = \prod_{k<k_F} \hat{d}_k^\dagger |0\rangle$$

(net momentum = 0)

In real space

Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**



Excited state w/ a particle

$$|k\rangle = \hat{d}^\dagger_{k>k_\mathrm{F}}|\mathrm{FS}\rangle$$

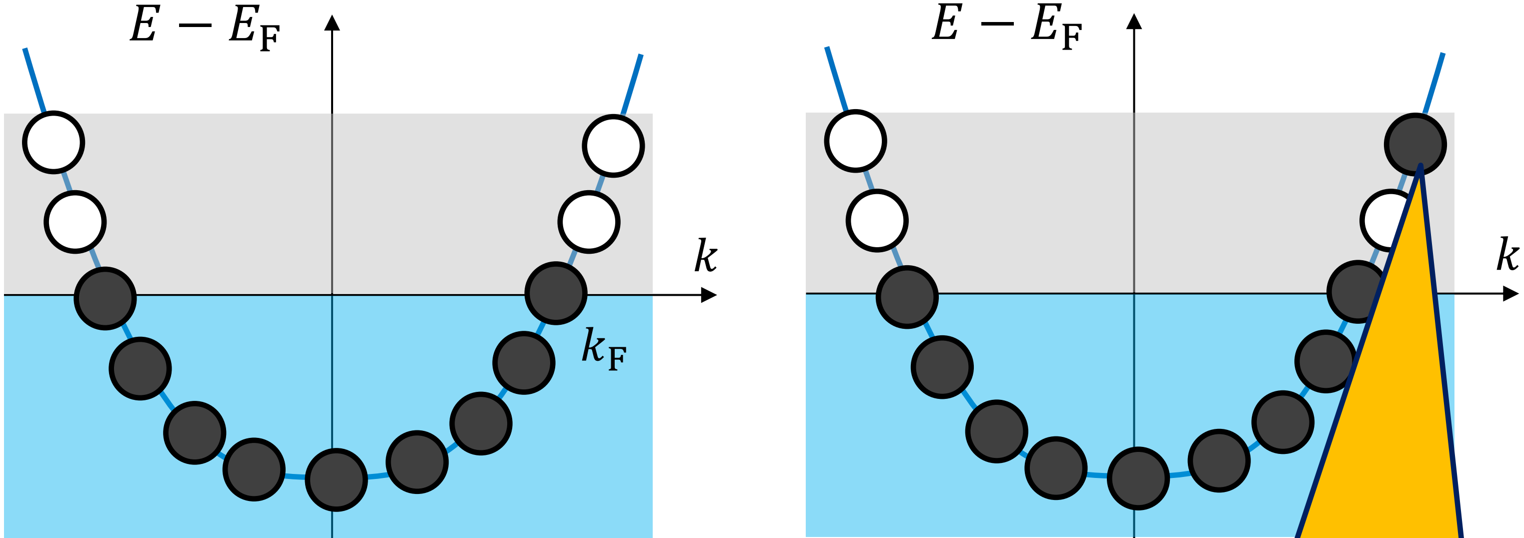
(net momentum = $\hbar k$)

In momentum space

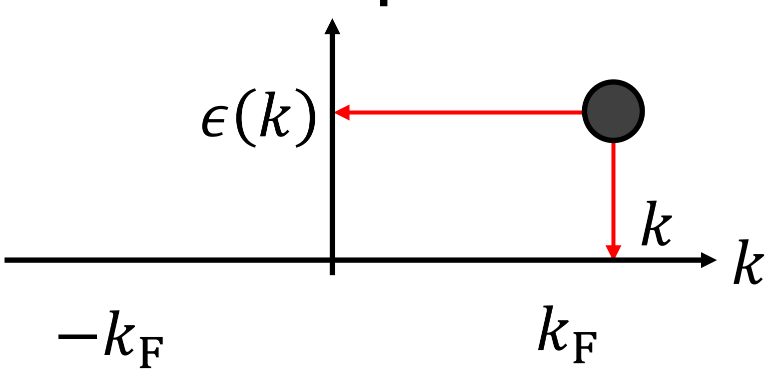
In real space

Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**



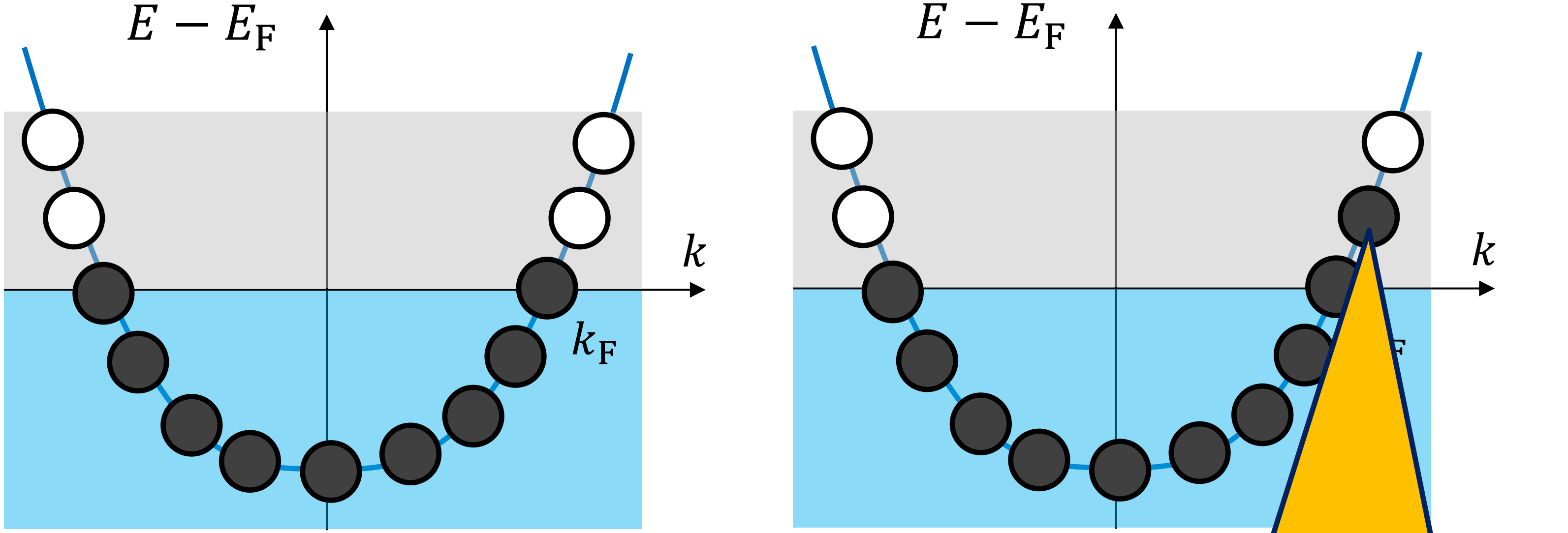
Excitation spectrum



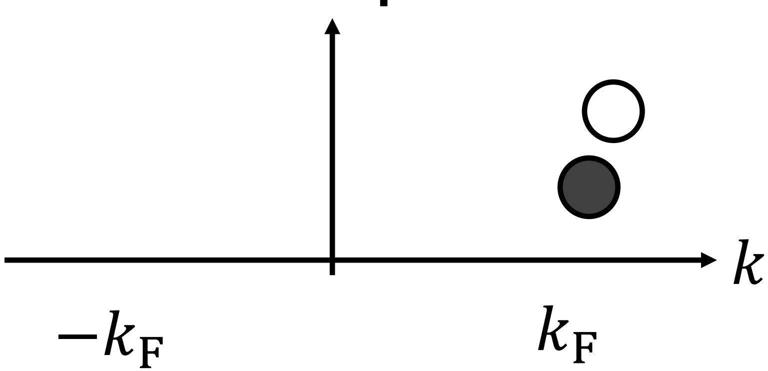
Excited state in energy by $\epsilon(k)$ & momentum by $\hbar k$

Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**



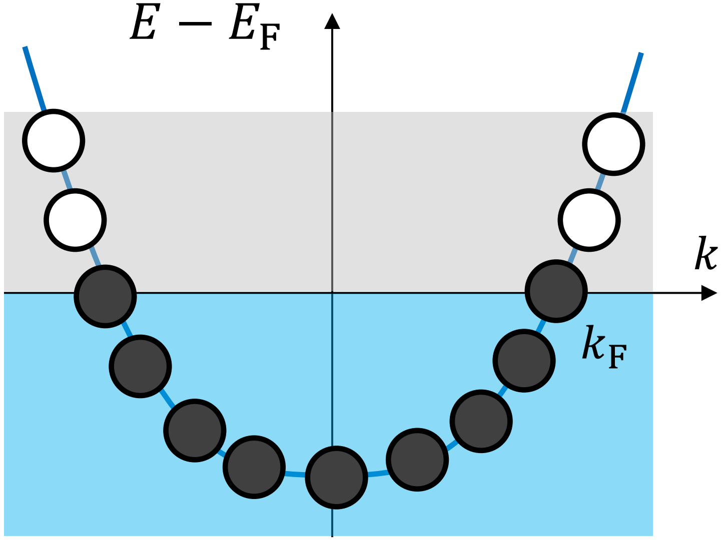
Excitation spectrum

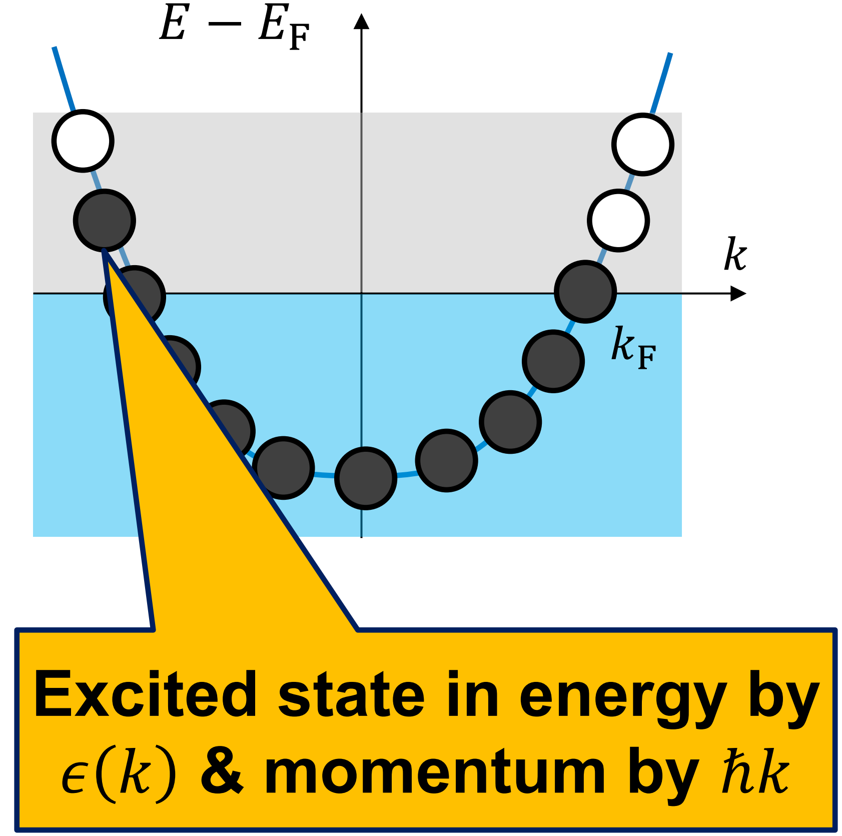


Excited state in energy by $\epsilon(k)$ & momentum by $\hbar k$

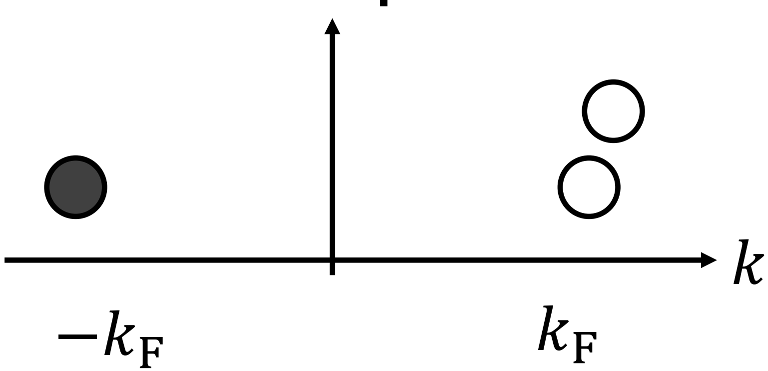
Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**



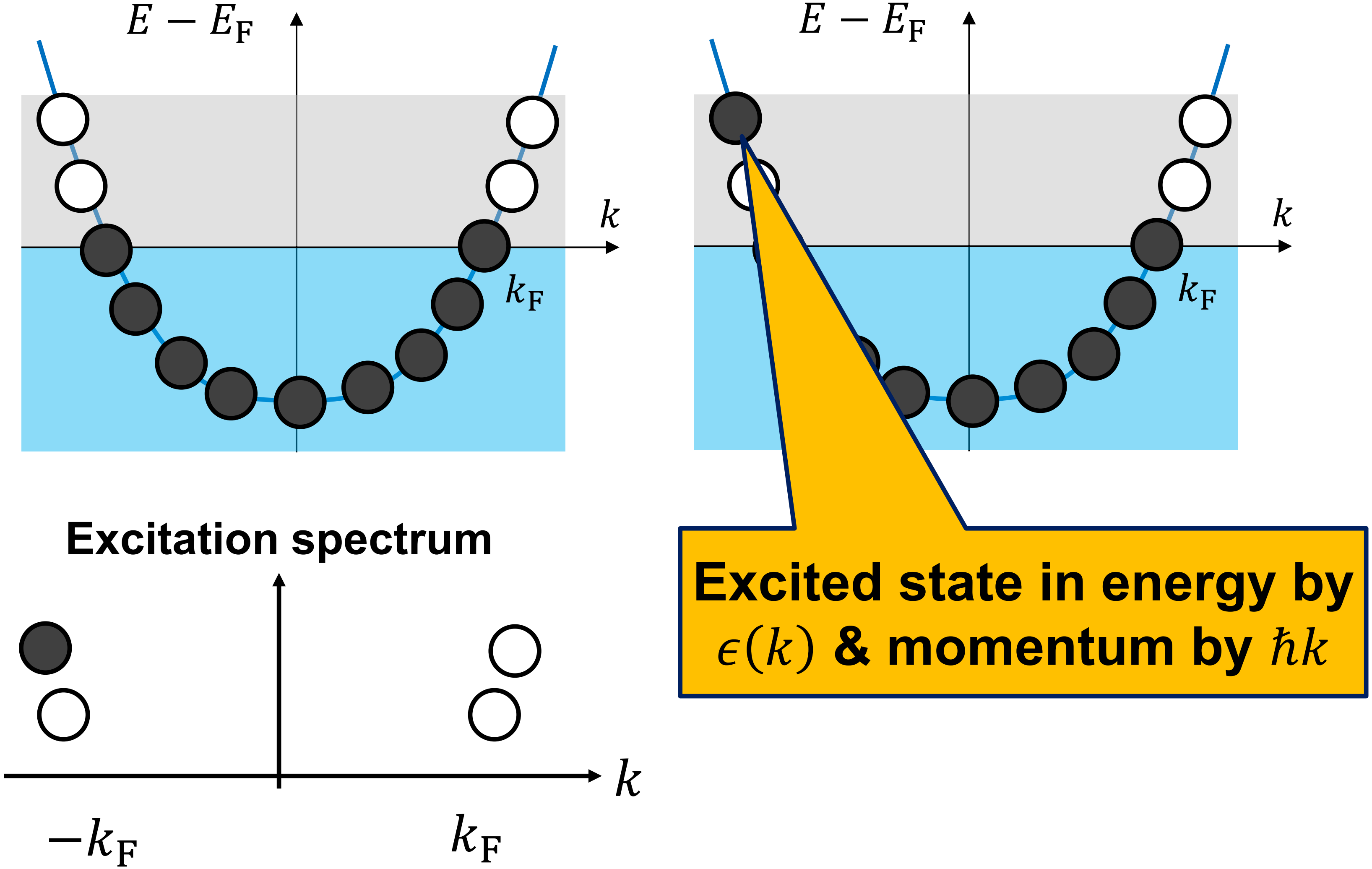


Excitation spectrum



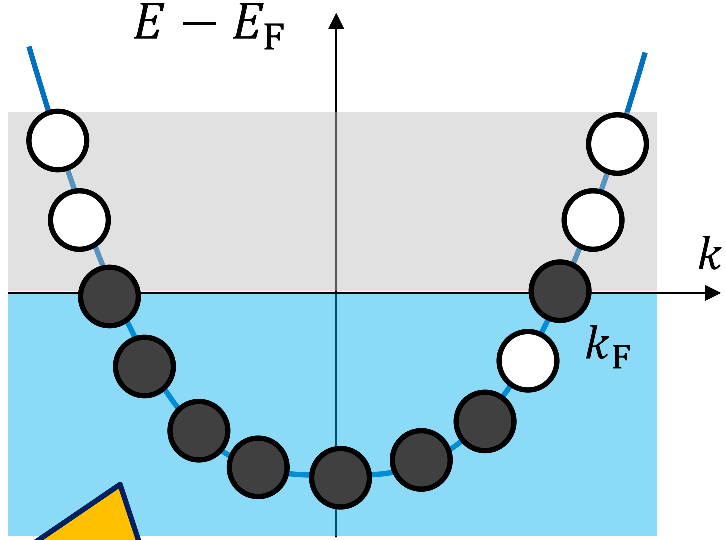
Route II:
mesoscopic superconductors from AR

- Particle & Hole excitations in normal metals



Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**



Excited state w/ a hold

$$|k\rangle = \hat{d}_{k<k_{\mathrm{F}}}|\mathrm{FS}\rangle$$

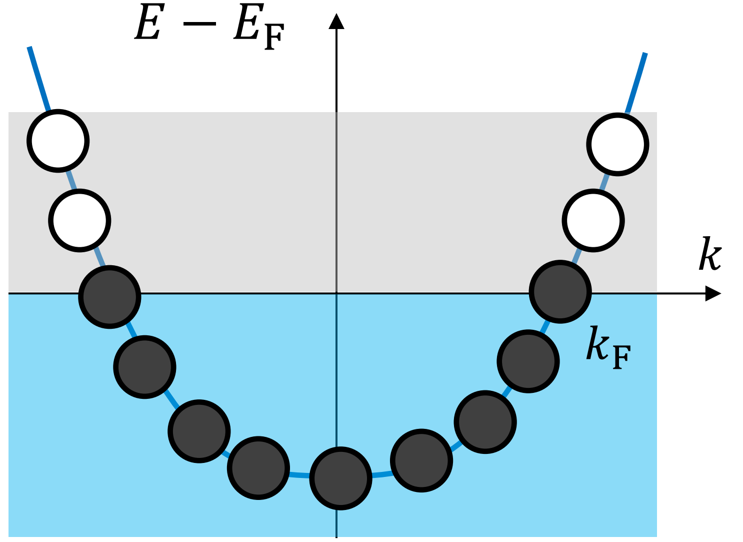
(net momentum = $-\hbar k$)

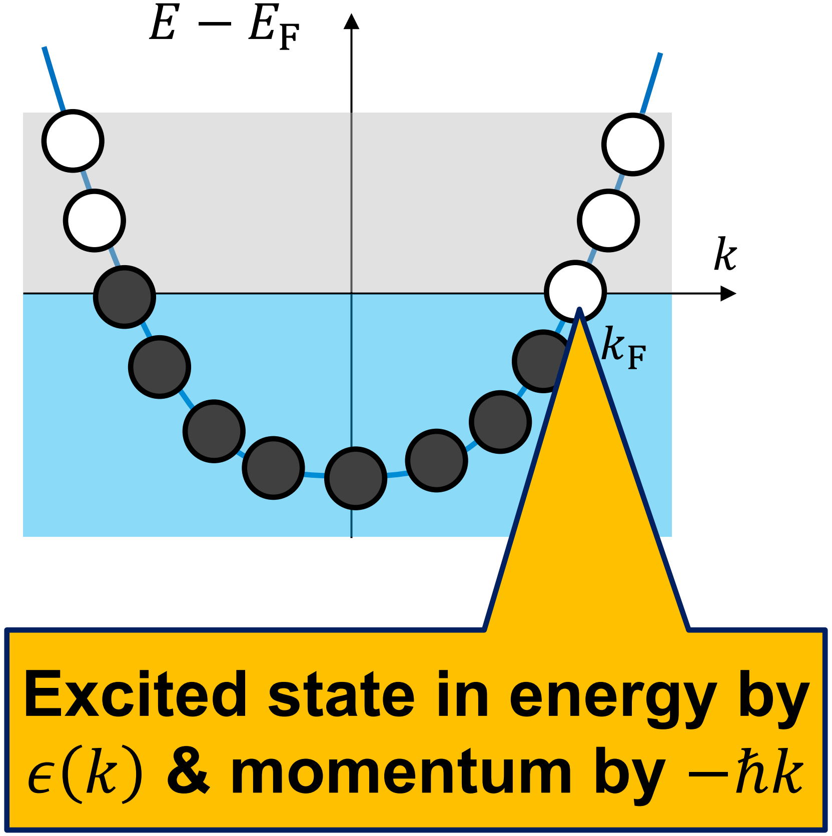
In momentum space

In real space

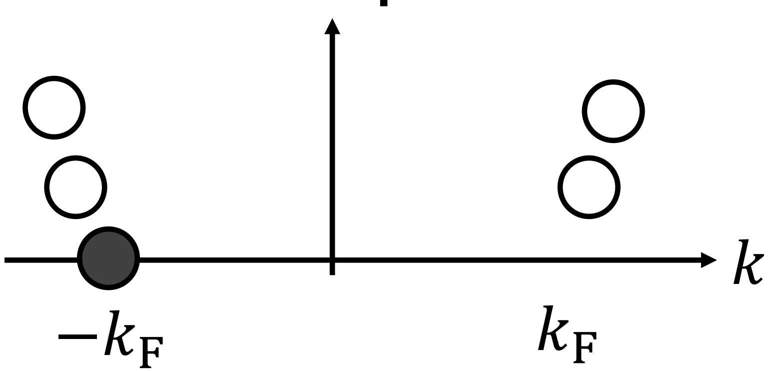
Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**



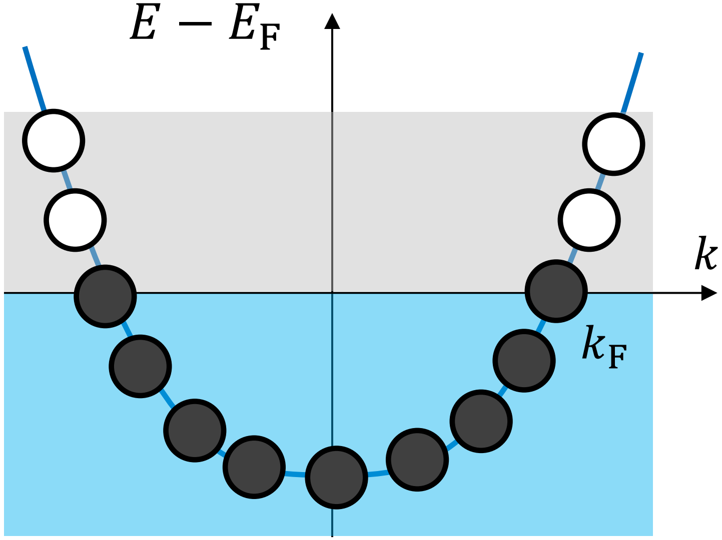


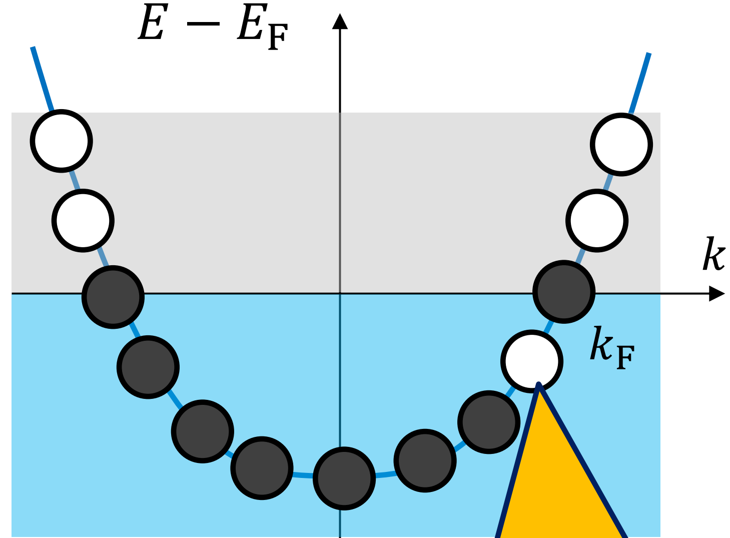
Excitation spectrum



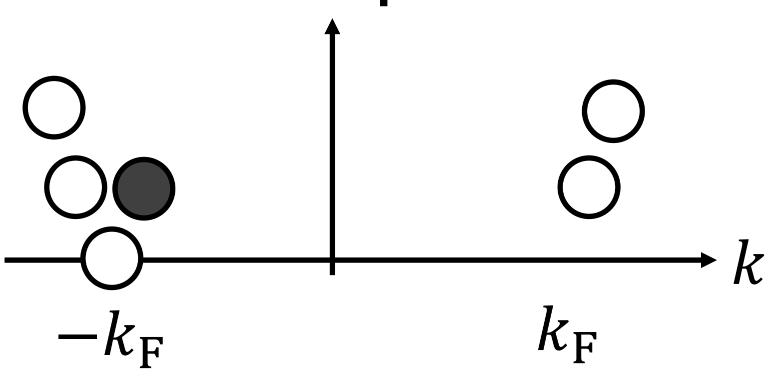
Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**





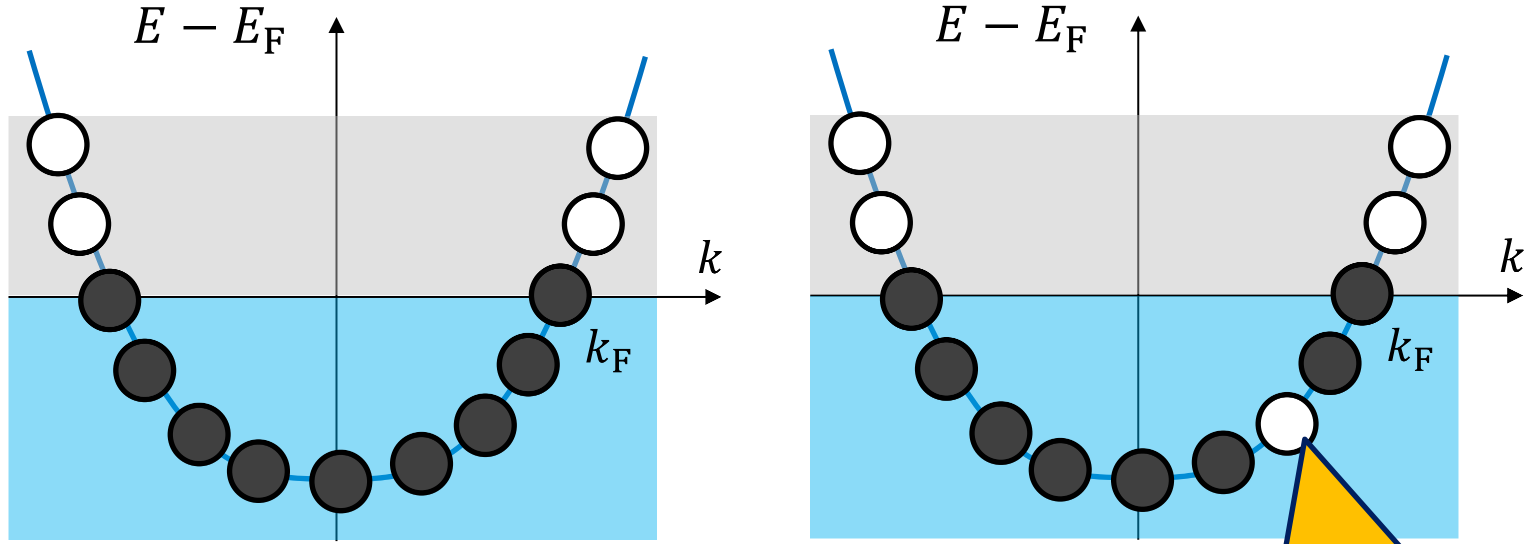
Excitation spectrum



Excited state in energy by $\epsilon(k)$ & momentum by $-\hbar k$

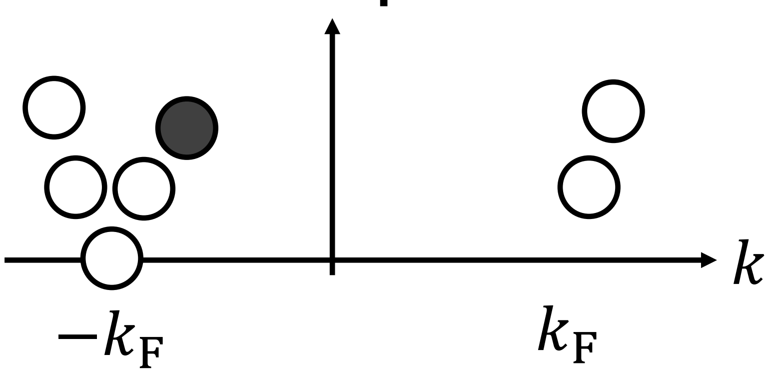
Route II: mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**



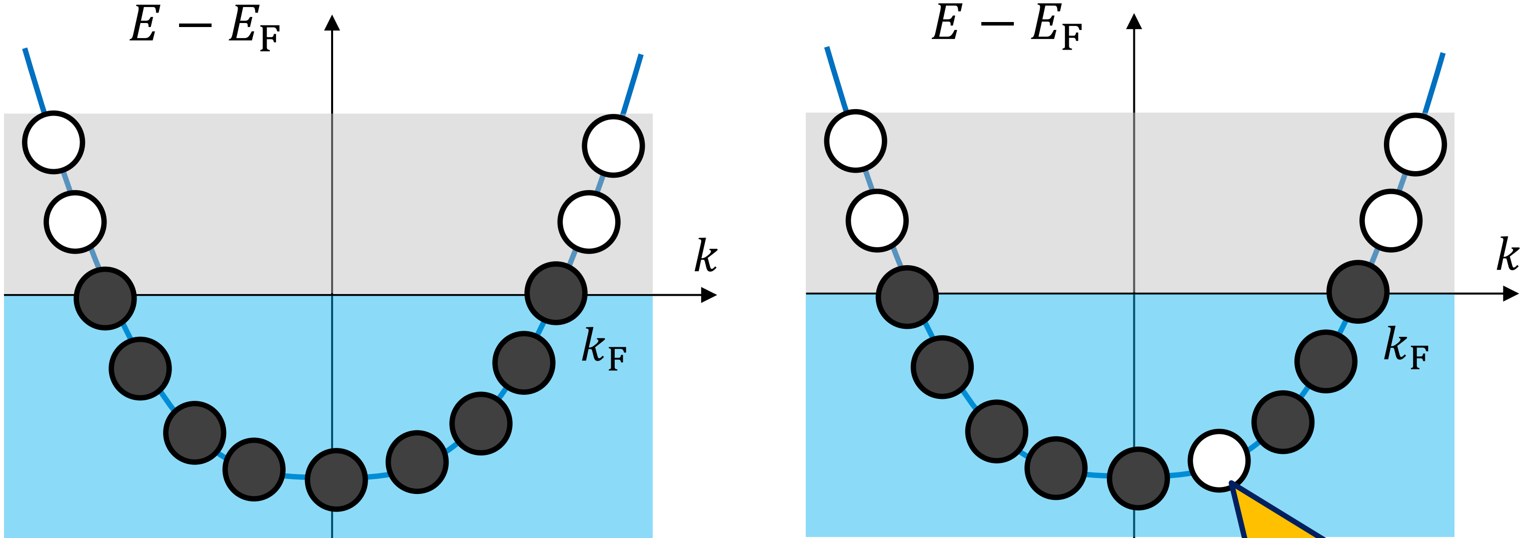
Excited state in energy by $\epsilon(k)$ & momentum by $-\hbar k$

Excitation spectrum



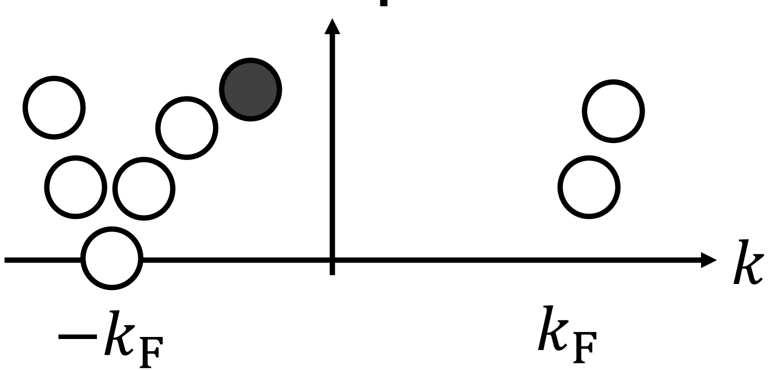
Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**



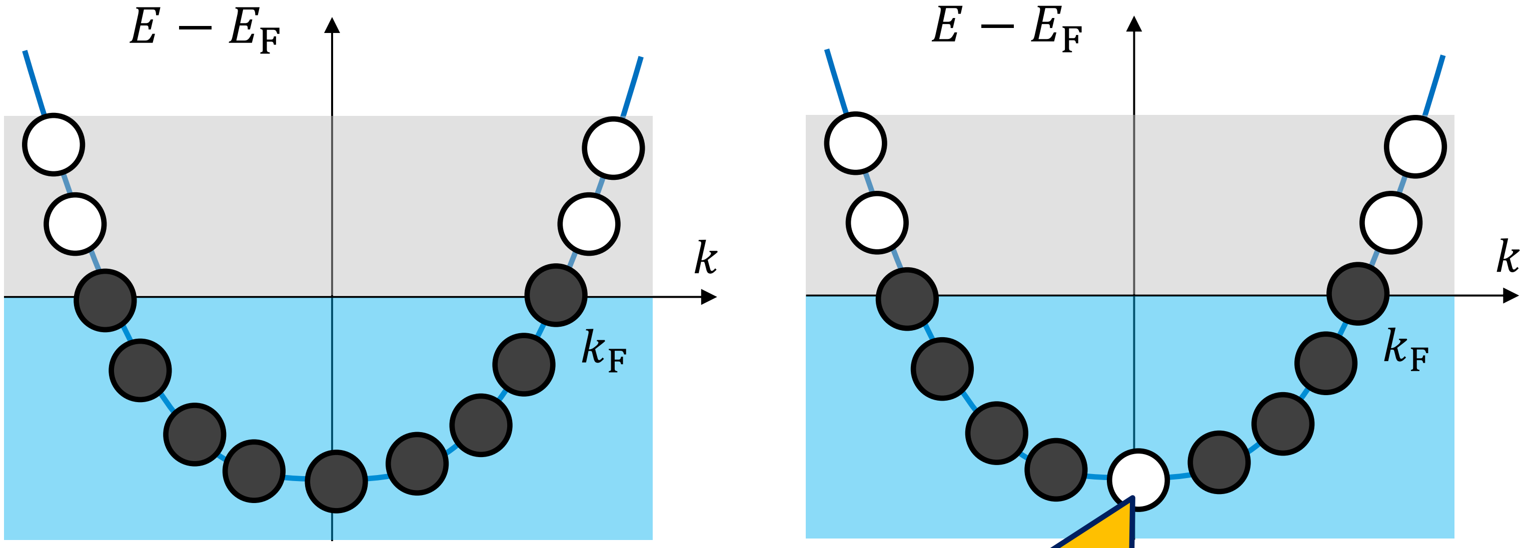
Excited state in energy by $\epsilon(k)$ & momentum by $-\hbar k$

Excitation spectrum

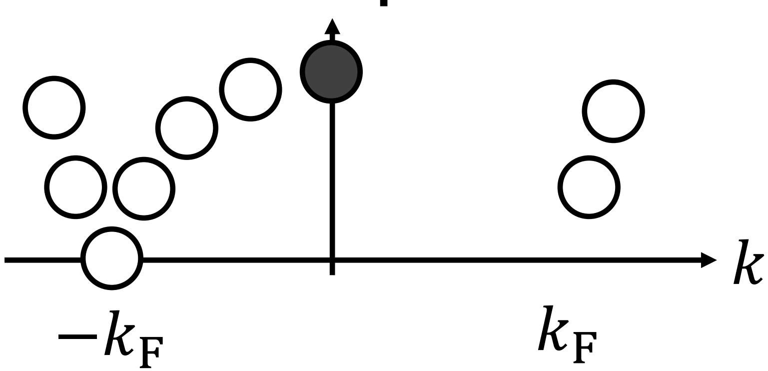


Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**



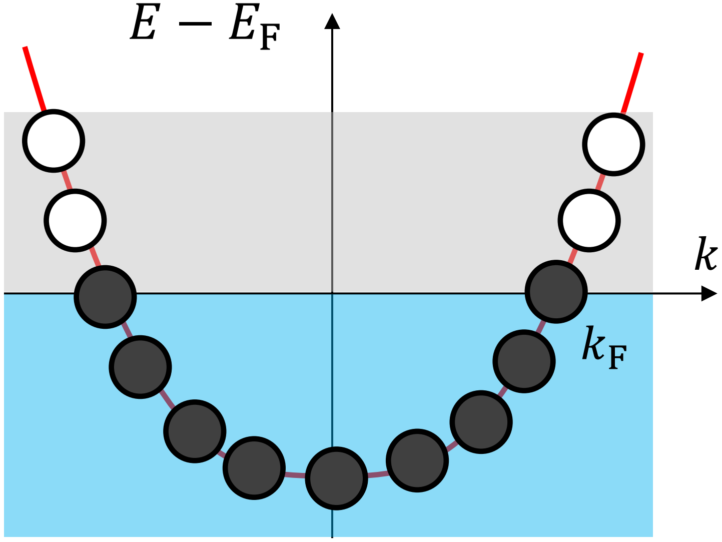
Excitation spectrum

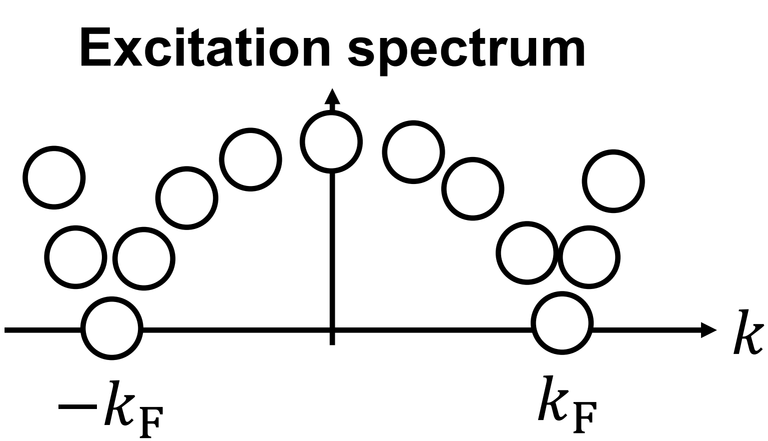


Excited state in energy by $\epsilon(k)$ & momentum by $-\hbar k$

Route II: mesoscopic superconductors from AR

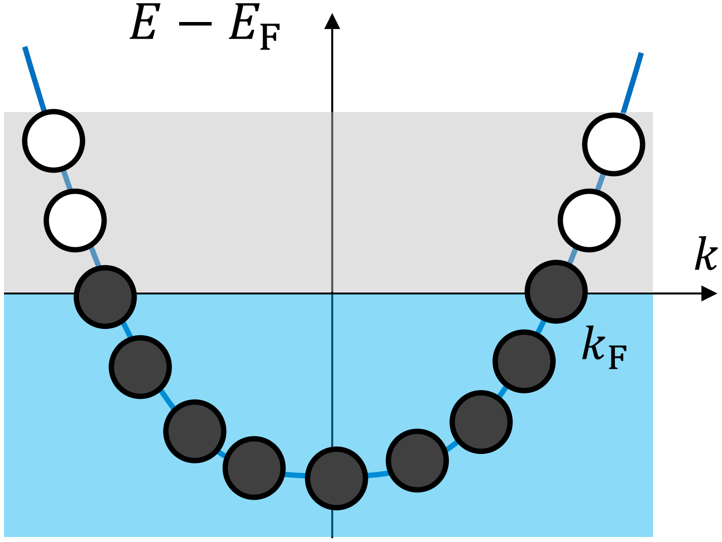
- Particle & Hole excitations in normal metals





Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**

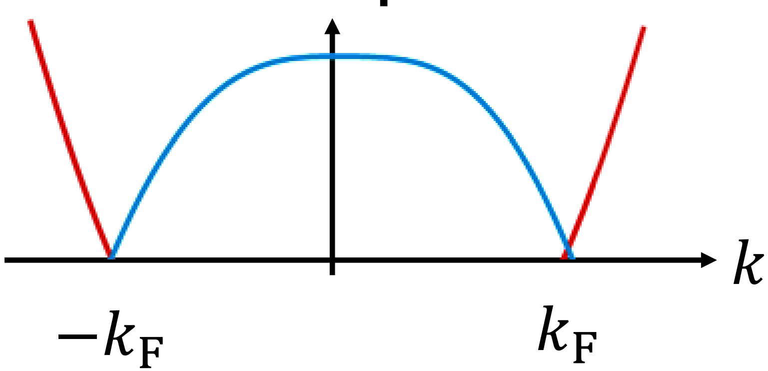


Ground state = Fermi sea (FS)

$$|\mathrm{FS}\rangle = \prod_{k<k_{\mathrm{F}}} \hat{d}_k^\dagger |0\rangle$$

FS = the state with a completely empty excitation spectrum

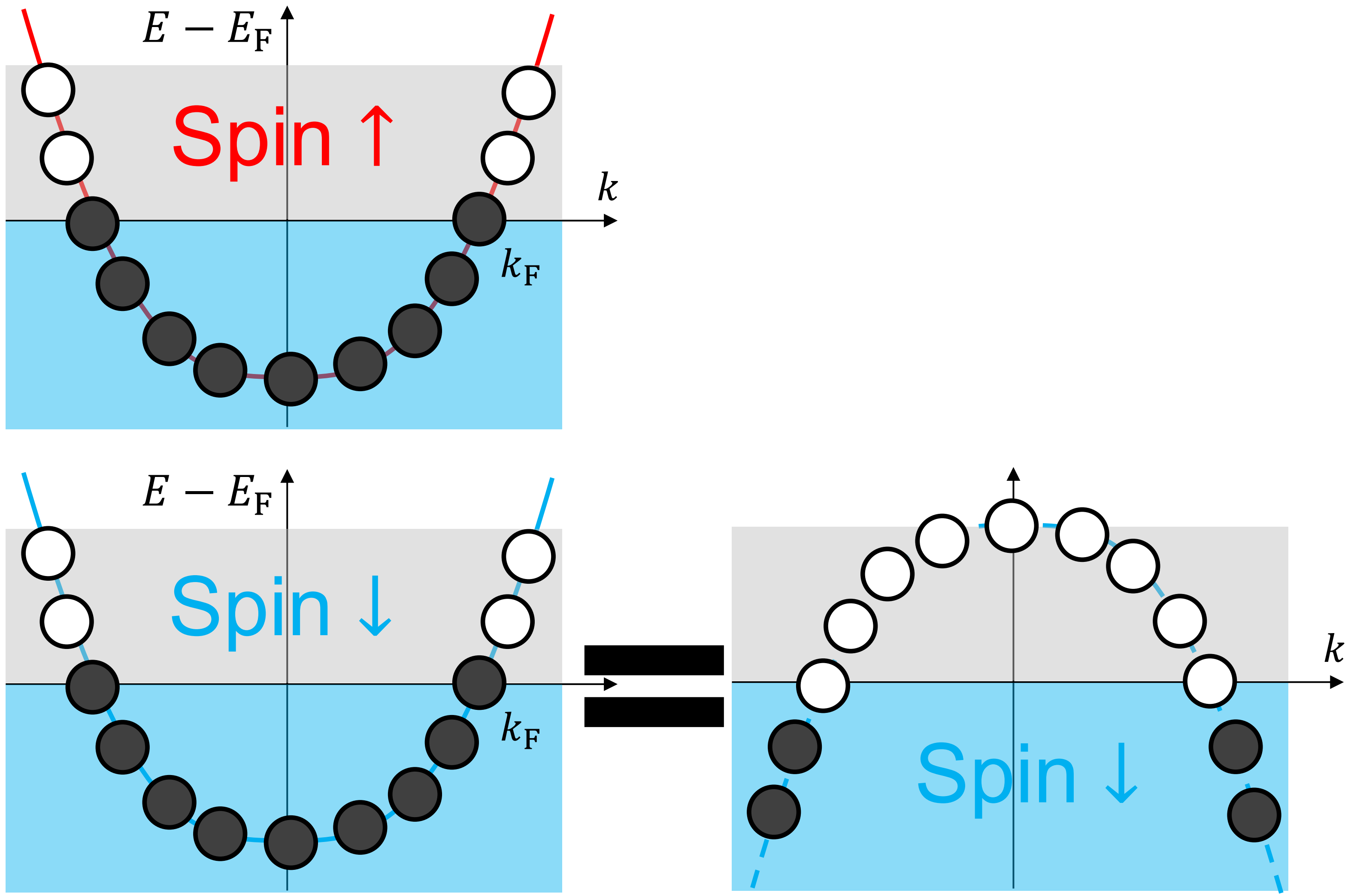
Excitation spectrum



Limited to a spinless case

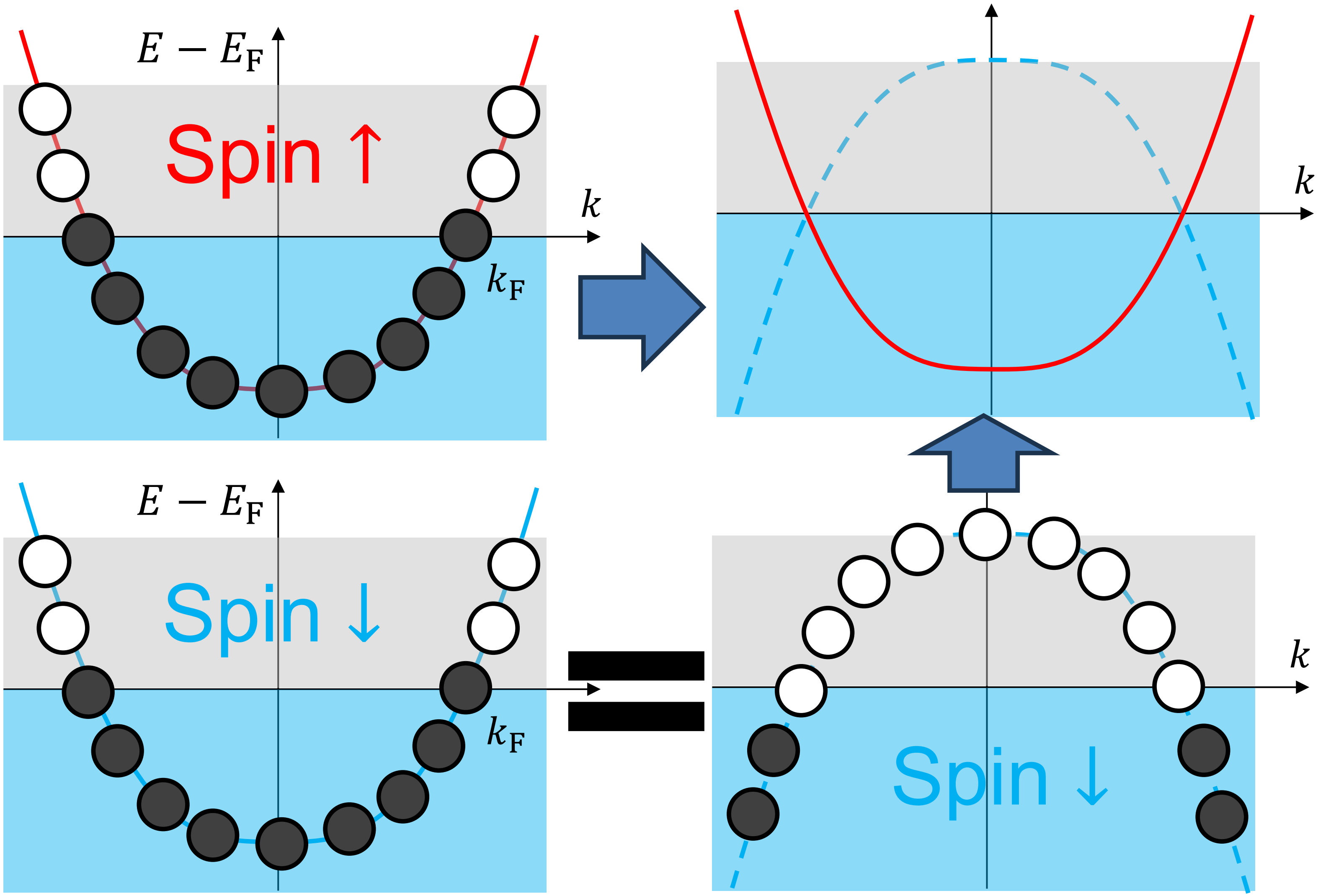
Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**



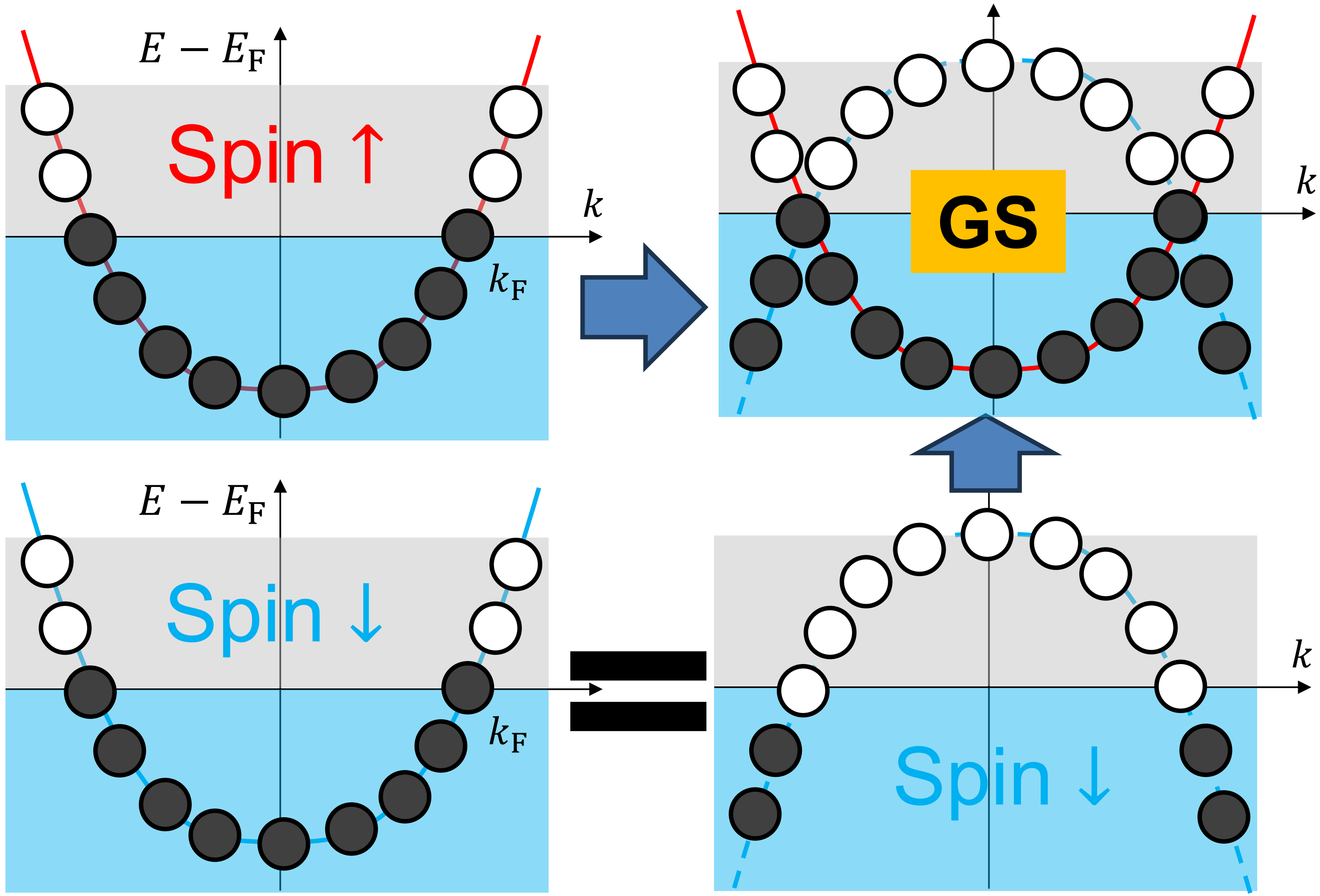
Route II:
mesoscopic superconductors from AR

- Particle & Hole excitations in normal metals



Route II:
mesoscopic superconductors from AR

- Particle & Hole excitations in normal metals

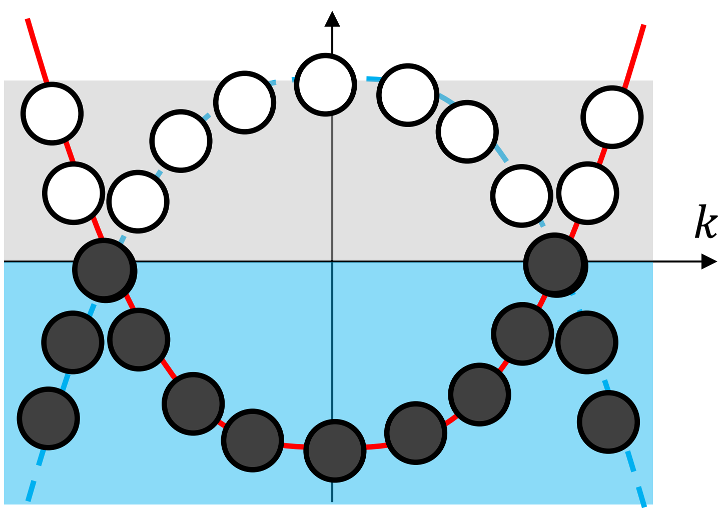


Route II:
mesoscopic superconductors from AR

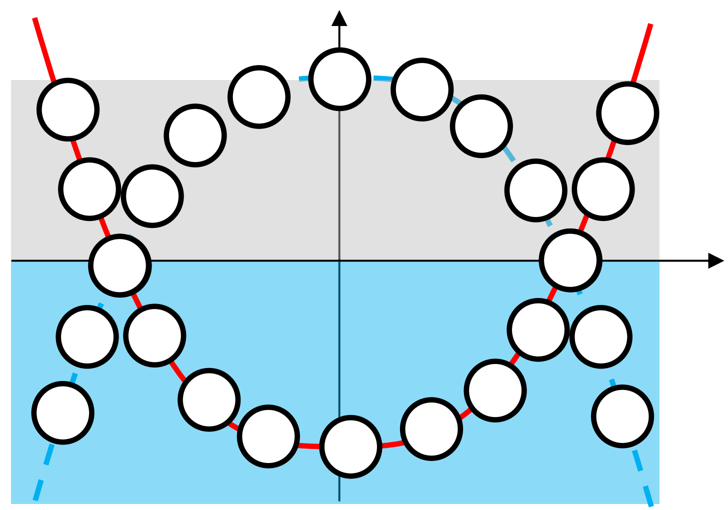
- **Particle & Hole excitations in normal metals**

  The Bogoliubov-de Genne (BdG) Hamiltonian

$$H = \begin{pmatrix} \epsilon(k) - E_\mathrm{F} & 0 \\ 0 & -\epsilon(-k) + E_\mathrm{F} \end{pmatrix}$$



GS

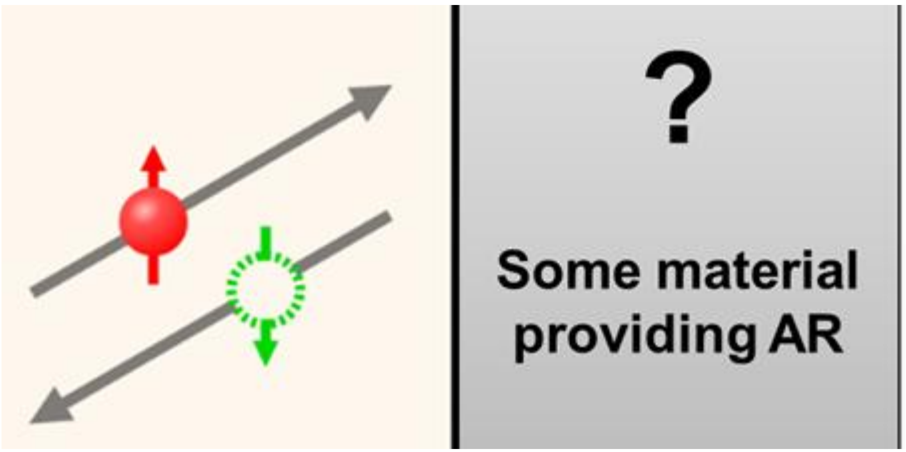


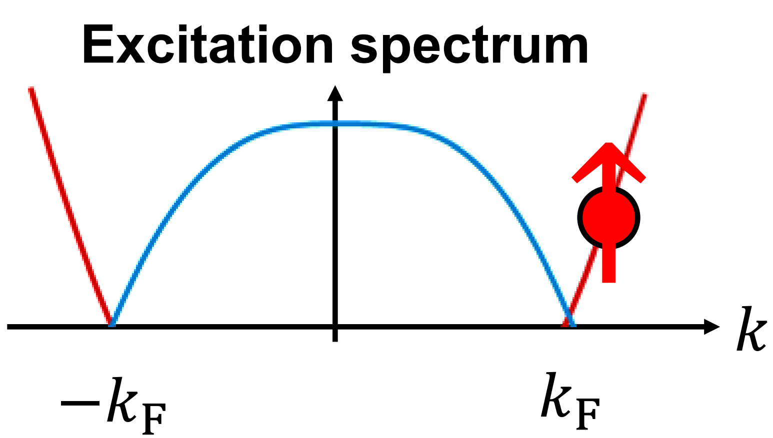
Vacuum of BdG Hamiltonian

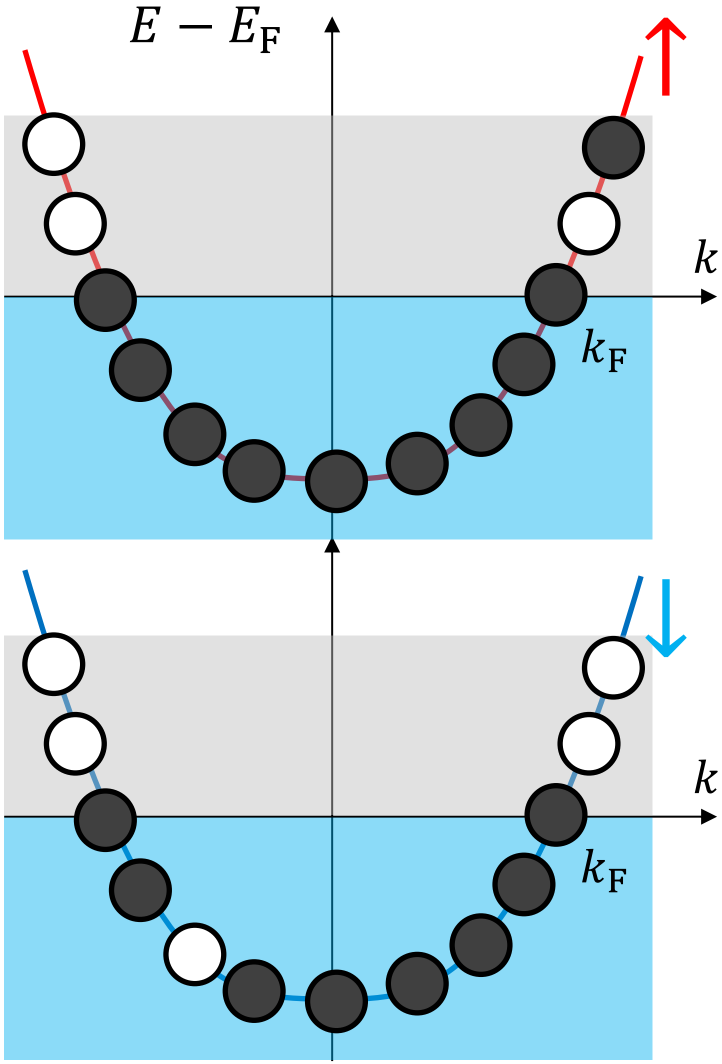
$|\mathrm{vac}\rangle = \prod_k \hat{d}^\dagger_{-k,\downarrow}|0\rangle$

Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**

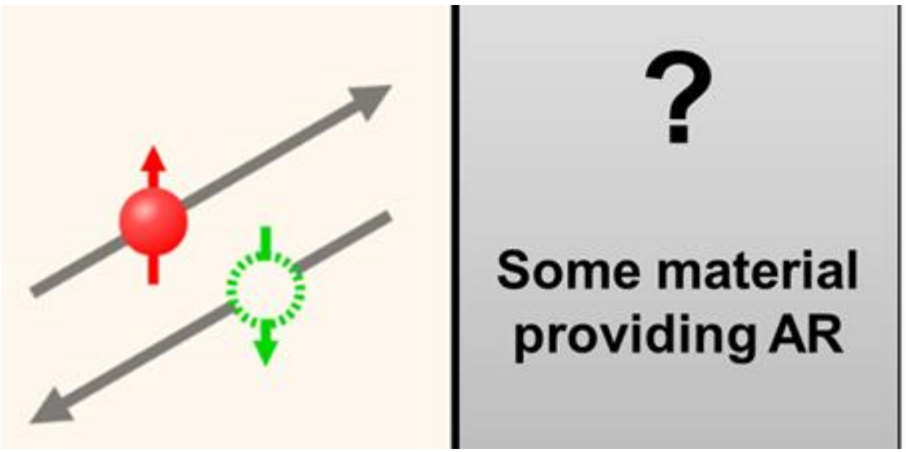


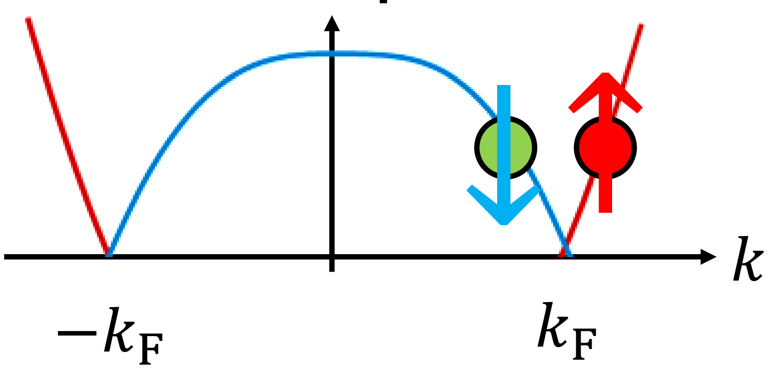


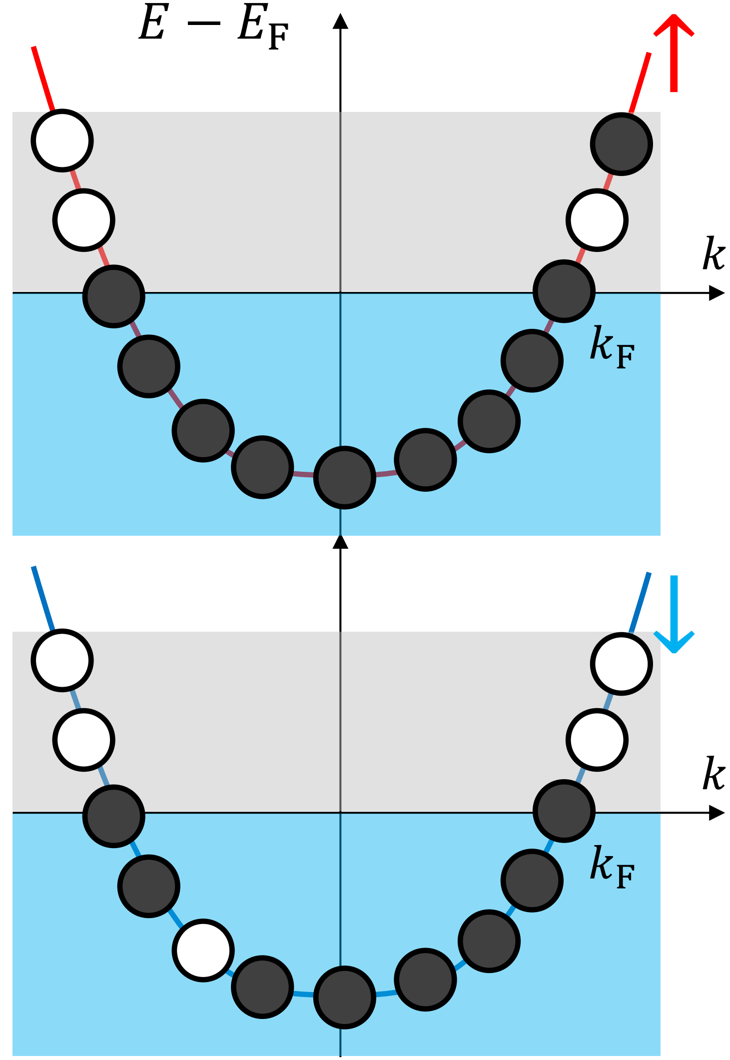


Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**

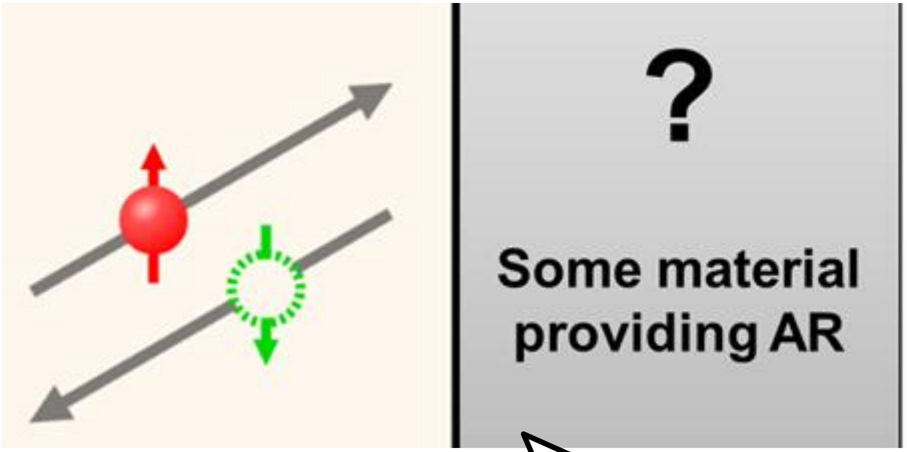






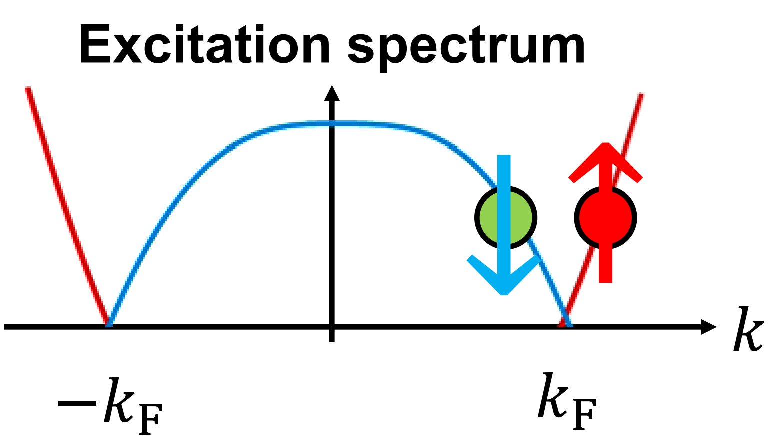
Route II:
mesoscopic superconductors from AR

- **Particle & Hole excitations in normal metals**



Hamiltonian should include e-h hybridization. Otherwise, AR is not allowed.

$$H = \begin{pmatrix} \epsilon(k) - E_{\mathrm{F}} & U \\ U^* & -\epsilon(-k) + E_{\mathrm{F}} \end{pmatrix}$$

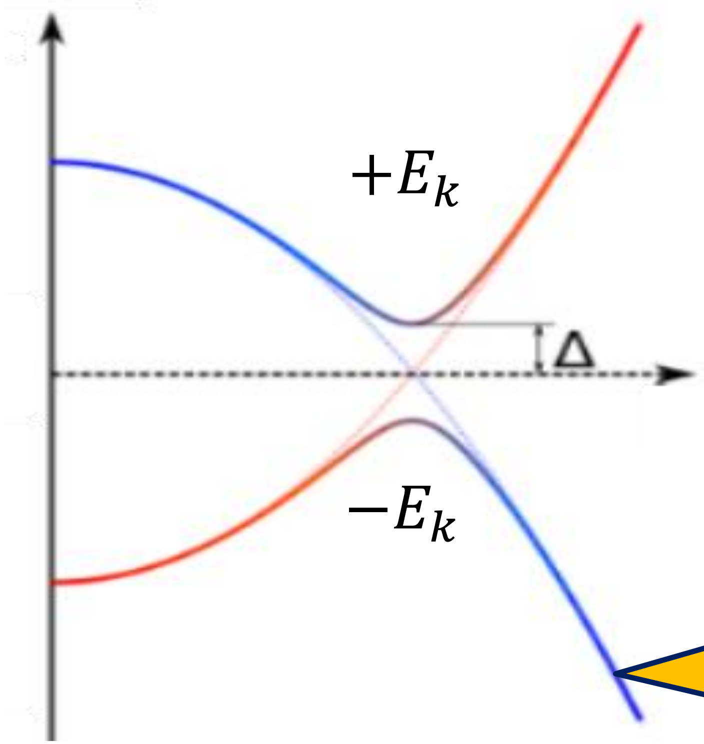


Route II:
mesoscopic superconductors from AR

- **A system with e-h hybridization**

$$H = \begin{pmatrix} \epsilon(k) - E_\mathrm{F} & U \\ U^* & -\epsilon(-k) + E_\mathrm{F} \end{pmatrix} = \begin{pmatrix} \xi_k & \Delta e^{i\varphi} \\ \Delta e^{-i\varphi} & -\xi_k \end{pmatrix}$$

→ Eigenenergies: $\pm E_k = \pm\sqrt{\xi_k^2 + \Delta^2}$



Ground state of a system with e-h hybridization = the state filling the spectrum $-E_k$

Route II: mesoscopic superconductors from AR

- **A system with e-h hybridization**

$$H = \begin{pmatrix} \epsilon(k) - E_\mathrm{F} & U \\ U^* & -\epsilon(-k) + E_\mathrm{F} \end{pmatrix} = \begin{pmatrix} \xi_k & \Delta e^{i\varphi} \\ \Delta e^{-i\varphi} & -\xi_k \end{pmatrix}$$

$\rightarrow$ Corresponding single-particle states

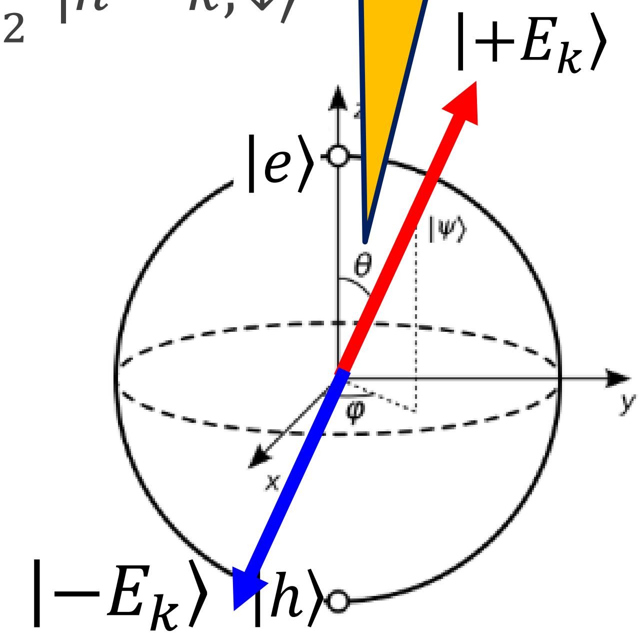
$$|+E_k\rangle = \cos\frac{\theta_k}{2}|e,k,\uparrow\rangle + e^{-i\varphi}\sin\frac{\theta_k}{2}|h,-k,\downarrow\rangle$$

$$|-E_k\rangle = -e^{i\varphi}\sin\frac{\theta_k}{2}|e,k,\uparrow\rangle + \cos\frac{\theta_k}{2}|h-k,\downarrow\rangle$$

$$\sin^2\frac{\theta_k}{2} = \frac{1}{2}(1-\cos\theta_k) = \frac{1}{2}\left(1-\frac{\xi_k}{E_k}\right)$$

$$\cos^2\frac{\theta_k}{2} = \frac{1}{2}(1+\cos\theta_k) = \frac{1}{2}\left(1+\frac{\xi_k}{E_k}\right)$$

$$\tan\theta_k = \frac{\Delta}{\xi_k}$$



Recall that

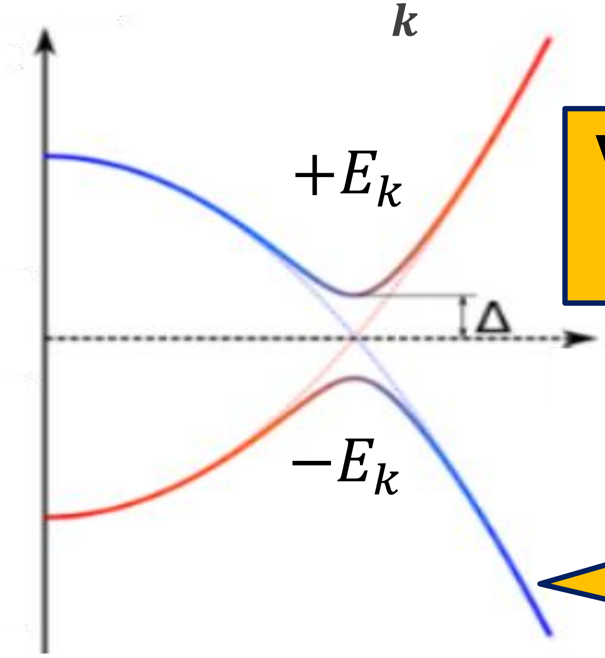
$$u_{\boldsymbol{k}}^2 = \frac{1}{2}\left(1+\frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right) \;\&\; v_{\boldsymbol{k}}^2 = \frac{1}{2}\left(1-\frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right)$$

Route II: mesoscopic superconductors from AR

- **The ground state of a system with e-h hybridization**
  $\longrightarrow$ the state filling the spectrum $-E_k$
  $|-E_k\rangle = -e^{i\varphi} v_{\boldsymbol{k}} |e, k, \uparrow\rangle + u_{\boldsymbol{k}} |h, -k, \downarrow\rangle$
  $\Leftrightarrow \hat{C}_{\boldsymbol{k}}^{\dagger} = u_{\boldsymbol{k}} \hat{d}_{-\boldsymbol{k},\downarrow} - e^{i\varphi} v_{\boldsymbol{k}} \hat{d}_{\boldsymbol{k},\uparrow}^{\dagger}$

$$|\mathrm{GS}\rangle = \prod_{\boldsymbol{k}} \hat{C}_{\boldsymbol{k}}^{\dagger} |\mathrm{vac}\rangle = \prod_{k} \left(u_{\boldsymbol{k}} \hat{d}_{-\boldsymbol{k},\downarrow} - e^{i\varphi} v_{\boldsymbol{k}} \hat{d}_{\boldsymbol{k},\uparrow}^{\dagger}\right) \hat{d}_{-\boldsymbol{k},\downarrow}^{\dagger} |0\rangle$$

Vac of BdG Hamiltonian
$|\mathrm{vac}\rangle = \prod_k \hat{d}_{-k,\downarrow}^{\dagger} |0\rangle$



Ground state of a system with e-h hybridization = the state filling the spectrum $-E_k$

Route II: mesoscopic superconductors from AR

- **The ground state of a system with e-h hybridization**
  $\longrightarrow$ the state filling the spectrum $-E_k$

$$|-E_k\rangle = -e^{i\varphi} v_{\boldsymbol{k}} |e, k, \uparrow\rangle + u_{\boldsymbol{k}} |h, -k, \downarrow\rangle$$

$$\Leftrightarrow \hat{C}_{\boldsymbol{k}}^\dagger = u_{\boldsymbol{k}} \hat{d}_{-\boldsymbol{k},\downarrow} - e^{i\varphi} v_{\boldsymbol{k}} \hat{d}_{\boldsymbol{k},\uparrow}^\dagger$$

$$|\mathrm{GS}\rangle = \prod_{\boldsymbol{k}} \hat{C}_{\boldsymbol{k}}^\dagger |\mathrm{vac}\rangle = \prod_{k} \left(u_{\boldsymbol{k}} \hat{d}_{-\boldsymbol{k},\downarrow} - e^{i\varphi} v_{\boldsymbol{k}} \hat{d}_{\boldsymbol{k},\uparrow}^\dagger\right) \hat{d}_{-\boldsymbol{k},\downarrow}^\dagger |0\rangle$$

$$= \prod_{\boldsymbol{k}} \left(u_{\boldsymbol{k}} \hat{d}_{-\boldsymbol{k},\downarrow} \hat{d}_{-\boldsymbol{k},\downarrow}^\dagger - e^{i\varphi} v_{\boldsymbol{k}} \hat{d}_{\boldsymbol{k},\uparrow}^\dagger \hat{d}_{-\boldsymbol{k},\downarrow}^\dagger\right) |0\rangle$$

$$= \prod_{\boldsymbol{k}} \left(u_{\boldsymbol{k}} - e^{i\varphi} v_{\boldsymbol{k}} \hat{d}_{\boldsymbol{k},\uparrow}^\dagger \hat{d}_{-\boldsymbol{k},\downarrow}^\dagger\right) |0\rangle$$

$$= \prod_{\boldsymbol{k}} \left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}} \hat{d}_{\boldsymbol{k},\uparrow}^\dagger \hat{d}_{-\boldsymbol{k},\downarrow}^\dagger\right) |0\rangle = |\mathrm{BCS}\rangle$$

We did $e^{i\varphi/2} \hat{d}_{\boldsymbol{k},\sigma}^\dagger \mapsto i \hat{d}_{\boldsymbol{k},\sigma}^\dagger$.
Recall that $|\mathrm{BCS}\rangle$
$= \prod_{\boldsymbol{k}} \left(|u_{\boldsymbol{k}}| + |v_{\boldsymbol{k}}| \hat{d}_{\boldsymbol{k},\uparrow}^\dagger \hat{d}_{-\boldsymbol{k},\downarrow}^\dagger\right) |0\rangle$

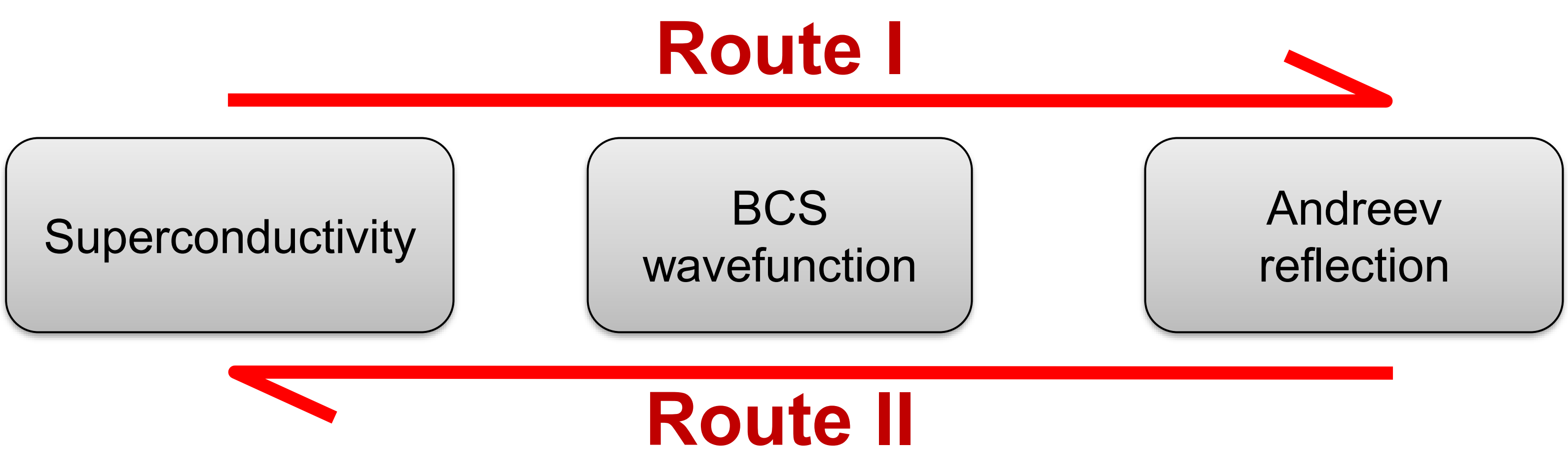
Route II:
mesoscopic superconductors from AR

- GS of a system with e-h hybridization
  ⇒ Superconducting BCS state

$$|\mathrm{GS}\rangle = \prod_{k} \hat{C}_{k}^{\dagger} |\mathrm{vac}\rangle = |\mathrm{BCS}\rangle$$

The message

There are two routes to understand SC & it means



Normal metals with AR ⇔ Superconductors

Route II:
mesoscopic superconductors from AR

- **Josephson junction & its theoretical descriptions**
  → Blonder-Tinkham-Klapwijk (BTK) model

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Transition from metallic to tunneling regimes in superconducting microconstrictions: Excess current, charge imbalance, and supercurrent conversion

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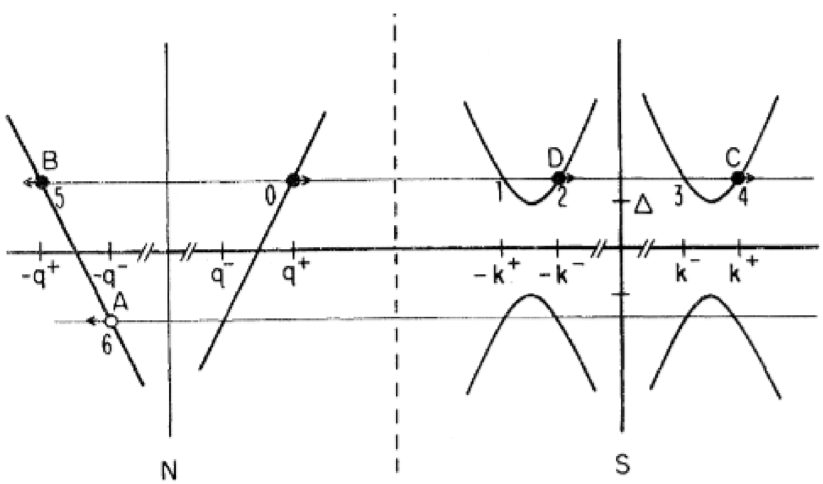
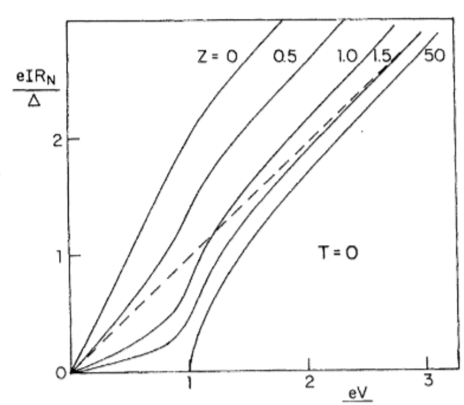
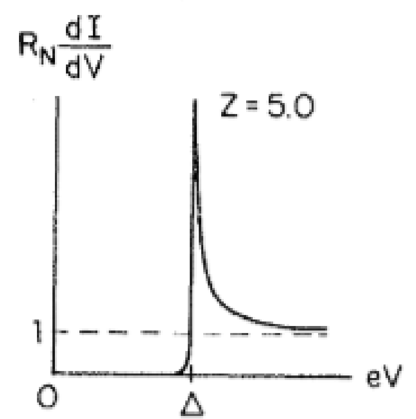
Physics Department, Harvard University, Cambridge, Massachusetts 02138
(Received 19 October 1981)



Excess current



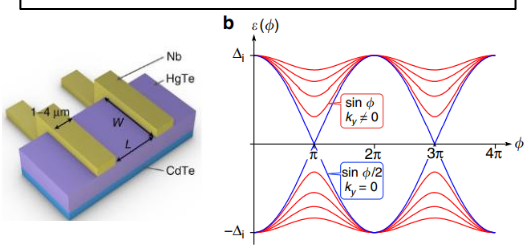
SC energy gap



Route II:
mesoscopic superconductors from AR

- **Josephson junction & its theoretical descriptions**
  → Blonder-Tinkham-Klapwijk (BTK) model
  → BTK is basically employing single-particle physics, yielding the same results to what employs the many-body BCS wavefunction!
  → It's extremely accessible and applicable to the wide range of cases (too good to be true, but it is!)

1. Free energy $F(\varphi)$



2. DC supercurrent $I_S(\varphi)$

$$I_S(\varphi) = \frac{2e}{\hbar}\frac{\partial F}{\partial \varphi}$$

3. Plug $I_S(\varphi)$ into RSJ solve EoM of $\varphi(t)$

$$I = I_S(\varphi) + \frac{\hbar}{2eR}\frac{d\varphi}{dt}$$

Voltage response $v(t)$ to current bias I

$$v(t) = \frac{\hbar}{2e}\frac{d\varphi}{dt}$$

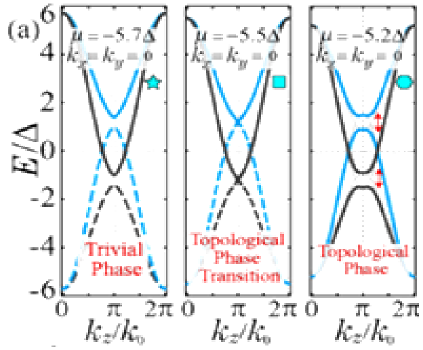
Fractional Josephson effect & Shapiro steps

$$I_S(\varphi) = I_{2\pi}\sin\varphi + I_{4\pi}\sin\frac{\varphi}{2}$$

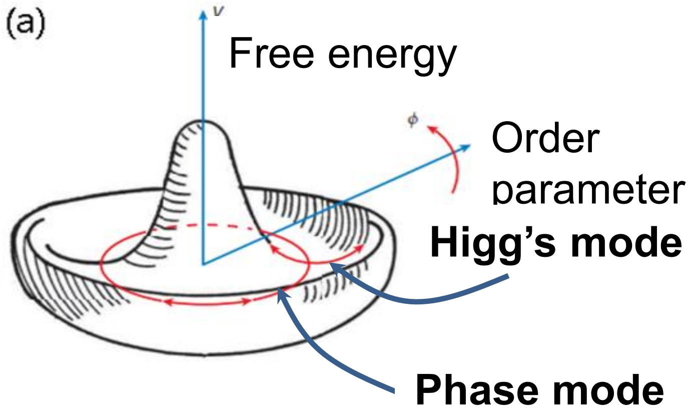
Beyond the conventional *s*-wave superconductivity

- **What there are under the carpet**
  → Multi-band SCs may not be simply BCS state
  → There are other pairing mechanisms. e.g., high-T_c
  → Omitted orbital effects, e.g., spin-3/2 + spin-1/2 band
  → Not directly applicable to strong el-ph coupling
  → Neglecting quantum fluctuations of Cooper pairs
  → Nonzero momentum SCs, e.g., FFLO, PDW
  → Competition against other phases such as CDW

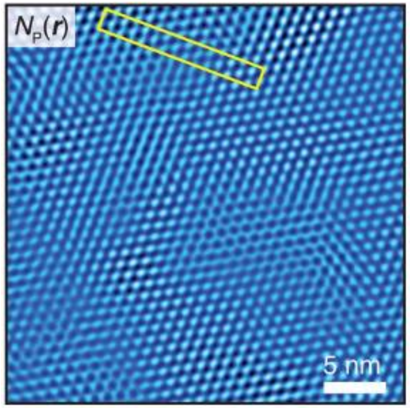
e.g., $Bi_2Te_3/NbSe_2$



M. Bahari, **S.-J. C**, et al., PRL (2024)



A. Lahiri, **S.-J. Choi**, et al., PRB (2024)



X. Liu, et al., Science (2021)