EM form factors of the three-nucleon systems in the Bethe-Salpeter-Faddeev approach

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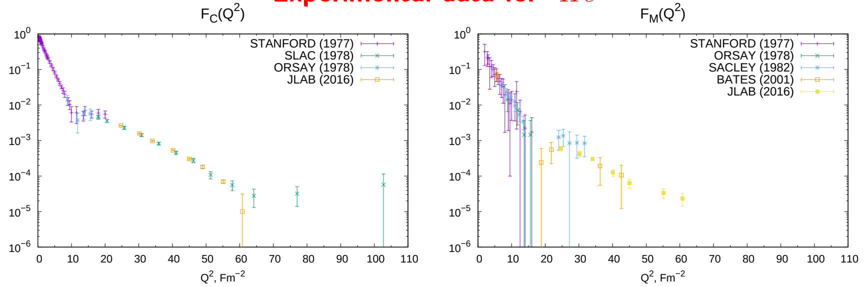
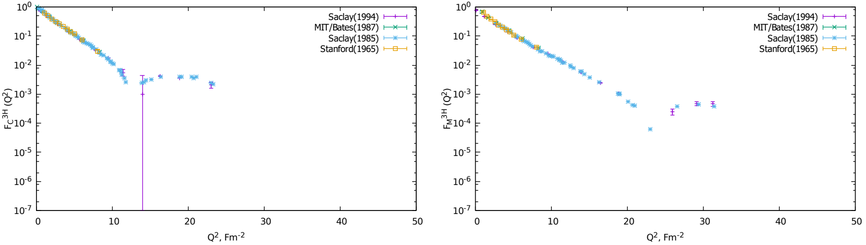
Motivation

- the relativistic properties of the Faddeev equation for a bound $3N$ system
- the dynamic relativistic properties of the reaction with a bound $3N$ system (EM form factors)
- $3N$ bound system: I=1/2 $\rightarrow$ two isobars $T =^3 H$ and ^3He; S=1/2 $\rightarrow$ two form factors F_C, F_M;

Experimental data for ^3He



Experimental data for ^3H



The relativistic three-particle equation for T matrix

is considered in the Faddeev form with the following assumptions:

- no three-particles interaction $V_{123} = \sum_{i \neq j} V_{ij}$
- two-particles interaction has the separable phenomenological form
- nucleon propagators are chosen in a scalar form
- the only strong interactions are considered (not EM), so $^3He \equiv T$

Bethe-Salpeter-Faddeev equation

$$\begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} - \begin{bmatrix} 0 & T_1 G_1 & T_1 G_1 \\ T_2 G_2 & 0 & T_2 G_2 \\ T_3 G_3 & T_3 G_3 & 0 \end{bmatrix} \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix},$$

where full three-particles T matrix $T = \sum_i T^{(i)}$, G_i is the free two-particles (j and n) Green function (ijn is cyclic permutation of (1,2,3)):

$$G_i(k_j, k_n) = 1/(k_j^2 - m_N^2 + i\epsilon)/(k_n^2 - m_N^2 + i\epsilon)$$

and T_i is the two-particles T matrix which can be written as follows

$$T_i(k_1, k_2, k_3; k_1', k_2', k_3') = (2\pi)^4 \delta^{(4)}(k_i - k_i') T_i(k_j, k_n; k_j', k_n')$$

with $s_i = (k_j + k_n)^2 = (k_j' + k_n')^2$.

Bethe-Salpeter equation for the nucleon-nucleon T matrix

$$T(p,p';P) = V(p,p';P) + \frac{i}{(2\pi)^4}\int d^4k\, V(p,k;P)\, G(k;P)\, T(k,p';P)$$

p', p - the relative four-momenta
P - the total four-momentum

$T(p,p';P)$ – two-nucleon t matrix
$V(p,p';P)$ – kernel of nucleon-nucleon interaction
$G(p;P)$ – free scalar two-particle propagator

$$G^{-1}(p;P) = \left[(P/2+p)^2 - m_N^2 + i\epsilon\right]\left[(P/2-p)^2 - m_N^2 + i\epsilon\right]$$

Separable kernels of the NN interaction

The separable kernels of the nucleon-nucleon interaction are widely used in the calculations. The separable kernel as a *nonlocal* covariant interaction representing complex nature of the space-time continuum.

Separable rank-one *Ansatz* for the kernel

$$V_L(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \lambda^{[L]}(s) g^{[L]}(p_0', |\mathbf{p}'|) g^{[L]}(p_0, |\mathbf{p}|)$$

Solution for the T matrix

$$T_L(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \tau(s)\, g^{[L]}(p_0', |\mathbf{p}'|)\, g^{[L]}(p_0, |\mathbf{p}|)$$

with

$$\Big[\tau(s)\Big]^{-1} = \Big[\lambda^{[L]}(s)\Big]^{-1} + h(s),$$

$$h(s) = \sum_{coupled\, L} h_L(s) = -\frac{i}{4\pi^3} \int dk_0 \int |\mathbf{k}|^2\, d|\mathbf{k}| \sum_L [g^{[L]}(k_0, |\mathbf{k}|)]^2 S(k_0, |\mathbf{k}|; s)$$

$g^{[L]}$ - the model function, $\lambda^{[L'L]}(s)$ - a model parameter.

The relativistic generalization of the NR Graz-II and Paris separable kernel:

- Graz-II: ${}^1S_0^+$ – rank 2, ${}^3S_1^+ - {}^3D_1$ – rank 3
- Paris-1,2: ${}^1S_0^+$ – rank 3, ${}^3S_1^+ - {}^3D_1$ – rank 4

Results for ${}^1S_0^+$ channel

	Exp.	Graz-II	Paris-1	Paris-2
a (fm)	-23.748	-23.77	-23.72	-23.72
r_0 (fm)	2.75	2.683	2.810	2.817

Results for ${}^3S_1^+ - {}^3D_1$ channels

	Exp.	Graz-II	Graz-II	Graz-II	Paris-1	Paris-2
p_d (%)		4	5	6	5.77	5.77
a (fm)	5.424	5.419	5.420	5.421	5.426	5.413
r_0 (fm)	1.759	1.780	1.779	1.778	1.775	1.765
E_d (MeV)	2.2246	2.2254	2.2254	2.2254	2.2246	2.2250

Partial-wave three-nucleon functions

$$\Psi^{(a)}_{\lambda L}(p_0,|\mathbf{p}|,q_0,|\mathbf{q}|;s) = g^{(a)}(p_0,|\mathbf{p}|)\tau^{(a)}[(\frac{2}{3}\sqrt{s}+q_0)^2-\mathbf{q}^2]\,\Phi^{(a)}_{\lambda L}(q_0,|\mathbf{q}|;s)$$

System of the integral equations

$$\Phi^{(a)}_{\lambda L}(q_0,|\mathbf{q}|;s) = \frac{i}{4\pi^3}\sum_{a'\lambda'}\int_{-\infty}^{\infty} dq'_0 \int_0^{\infty} \mathbf{q}'^2 d|\mathbf{q}'|\, Z^{(aa')}_{\lambda\lambda'}(q_0,q;q'_0,|\mathbf{q}'|;s)$$

$$\frac{\tau^{(a')}[(\frac{2}{3}\sqrt{s}+q'_0)^2-\mathbf{q}'^2]}{(\frac{1}{3}\sqrt{s}-q'_0)^2-\mathbf{q}'^2-m^2+i\epsilon}\Phi^{(a')}_{\lambda' L}(q'_0,|\mathbf{q}'|;s)$$

with effective kernels of equation

$$Z^{(aa')}_{\lambda\lambda'}(q_0,|\mathbf{q}|;q'_0,|\mathbf{q}'|;s) = C_{(aa')}\int d\cos\vartheta_{\mathbf{q}\mathbf{q}'}\,K^{(aa')}_{\lambda\lambda' L}(|\mathbf{q}|,|\mathbf{q}'|,\cos\vartheta_{\mathbf{q}\mathbf{q}'})$$

$$\frac{g^{(a)}(-q_0/2-q'_0,|\mathbf{q}/2+\mathbf{q}'|)g^{(a')}(q_0+q'_0/2,|\mathbf{q}+\mathbf{q}'/2|)}{(\frac{1}{3}\sqrt{s}+q_0+q'_0)^2-(\mathbf{q}+\mathbf{q}')^2-m_N^2+i\epsilon}$$

Singularities

Poles from one-particle propagator

$$q^{0\prime}_{1,2} = \frac{1}{3}\sqrt{s} \mp [E_{|\mathbf{q}'|} - i\epsilon]$$

Poles from propagator in Z-function

$$q^{0\prime}_{3,4} = -\frac{1}{3}\sqrt{s} - q^0 \pm [E_{|\mathbf{q}'+\mathbf{q}|} - i\epsilon]$$

Poles from Yamaguchi-functions

$$q^{0\prime}_{5,6} = -2q^0 \pm 2[E_{|\frac{1}{2}\mathbf{q}'+\mathbf{q}|,\beta} - i\epsilon]$$

and

$$q^{0\prime}_{7,8} = -\frac{1}{2}q^0 \pm \frac{1}{2}[E_{|\mathbf{q}'+\frac{1}{2}\mathbf{q}|,\beta} - i\epsilon]$$

Cuts from two-particle propagator τ

$$q^{0\prime}_{9,10} = \pm\sqrt{q'^2 + 4m^2} - \frac{2}{3}\sqrt{s} \qquad \text{and} \qquad \pm\infty$$

Poles from two-particle propagator τ

$$q^{0\prime}_{11,12} = \pm\sqrt{q'^2 + 4M_d^2} - \frac{2}{3}\sqrt{s}$$

Method of solution

- Wick-rotation procedure: $q_0 \to iq_4$
- The Gaussian quadrature with $N_1 \times N_2[q_4 \times |\mathbf{q}|]$ grid

$$q_4 = (1+x)/(1-x)$$

$$|\mathbf{q}| = (1+y)/(1-y)$$

- Iteration method to obtain the triton binding energy

$$\lim_{n\to\infty} \frac{\Phi_n(s)}{\Phi_{n-1}(s)}\Big|_{s=M_B^2} = 1$$

Triton binding energy (MeV)

Graz-II 4	8.628
Graz-II 5	8.223
Graz-II 6	7.832
Paris-1	7.545
Exp.	8.48

Electromagnetic form factors of three-nucleon systems:

$$2F_{\rm C}(^3\text{He}) = (2F^p_{\rm C} + F^n_{\rm C})F_1 - \frac{2}{3}(F^p_{\rm C} - F^n_{\rm C})F_2,$$

$$F_C(^3\text{H}) = (2F^n_{\rm C} + F^p_{\rm C})F_1 + \frac{2}{3}(F^p_{\rm C} - F^n_{\rm C})F_2,$$

$$\mu(^3\text{He})F_{\rm M}(^3\text{He}) = \mu_n F^n_{\rm M} F_1 + \frac{2}{3}(\mu_n F^n_{\rm M} + \mu_p F^p_{\rm M})F_2 + \frac{4}{3}(F^p_{\rm M} - F^n_{\rm M})F_3,$$

$$\mu(^3\text{H})F_{\rm M}(^3\text{H}) = \mu_p F^p_{\rm M} F_1 + \frac{2}{3}(\mu_n F^n_{\rm M} + \mu_p F^p_{\rm M})F_2 + \frac{4}{3}(F^n_{\rm M} - F^p_{\rm M})F_3,$$

Electric and magnetic form factors of the proton and neutron $F^{p,n}_{\rm C,M}$.

Impulse approximation:

$$F_i(Q) = \int d^4p \int d^4q \;\; G'_1(k'_1)\, G_1(k_1)\, G_2(k_2)\, G_3(k_3)\, f_i(p,q,q';P,P')$$

Nucleon propagators:

$$G_i(k_1) = \left[k_i^2 - m_N^2 + i\epsilon\right]^{-1}$$

$$G'_1(q'_0, q') = \left[(\frac{1}{3}\sqrt{s} - q'_0)^2 - \mathbf{q}'^2 - m_N^2 + i\epsilon\right]^{-1}$$

Three-nucleon vertex functions:

$$f_1 = \sum_{i=1}^{3} \Psi_i^*(p,q;P)\, \Psi_i(p,q';P')$$

$$f_2 = -3\Psi_1^*(p,q;P)\, \Psi_2(p,q';P')$$

$$f_3 = \Psi_3^*(p,q;P)\, \Psi_3(p,q';P')$$

Functions Ψ_i are the definite combinations of the partial state functions.

The Breit reference system

$$Q = (0, \mathbf{Q}), \qquad P = (E_B, -\frac{\mathbf{Q}}{2}), \qquad P' = (E_B, \frac{\mathbf{Q}}{2}), \tag{1}$$

with $E_B = \sqrt{\mathbf{Q}^2/4 + s}$, $\eta = \mathbf{Q}^2/4s$, $s = M_{3N}^2$.

$$\begin{array}{lll} P = \mathrm{L} P_{c.m.}, & p = \mathrm{L} p_{c.m.}, & q = \mathrm{L} q_{c.m.} \\ P' = \mathrm{L}^{-1} P'_{c.m.}, & p' = \mathrm{L}^{-1} p'_{c.m.}, & q' = \mathrm{L}^{-1} q'_{c.m.} \end{array}$$

The explicit form of the transformation L can be obtained by using (1). Let us assume the boost of the system to be along the Z axis:

$$\mathrm{L} = \begin{pmatrix} \sqrt{1+\eta} & 0 & 0 & -\sqrt{\eta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sqrt{\eta} & 0 & 0 & \sqrt{1+\eta} \end{pmatrix}. \tag{2}$$

Relation of the arguments of initial and final $3N$ functions:

$$
\begin{aligned}
q'_0 &= (1+2\eta)\, q_0 - 2\sqrt{\eta}\sqrt{1+\eta}\, q_z + \frac{2}{3}\sqrt{\eta}\, Q, \\
q'_x &= q_x \qquad q'_y = q_y \\
q'_z &= (1+2\eta)\, q_z - 2\sqrt{\eta}\sqrt{1+\eta}\, q_0 - \frac{2}{3}\sqrt{1+\eta}\, Q,
\end{aligned} \tag{3}
$$

here $q_z = q\cos\theta_{qQ}$ is the projection of momentum $\mathbf{q}$ onto the Z axis

Static approximation (SA):

$$q_0' = q_0, \qquad \mathbf{q}' = \mathbf{q} - \frac{2}{3}\mathbf{Q}$$

Propagator and final function:

$$G_1'(q_0', q') \to \left[(\frac{1}{3}\sqrt{s} - q_0)^2 - \mathbf{q}^2 - \frac{2}{3}\mathbf{q}\cdot\mathbf{Q} - \frac{4}{9}\mathbf{Q}^2 - m_N^2 + i\epsilon \right]^{-1}$$

$$\Psi_i(p_0, p, q_0', q') \to \Psi_i(p_0, p, q_0, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|)$$

with $\mathbf{q}\cdot\mathbf{Q} = qQ\cos\theta_{qQ}$.
The poles of G_1' on q_0 do not cross the imaginary q_0 axis and always stay in the second and fourth quadrants. In this case, the Wick rotation procedure $q_0 \to iq_4$ can be applied.

Beyond the SA:

1. <u>Exact propagator</u>

$$G_1' = \left[q_0^2 + \frac{2}{3}\sqrt{s}(1+6\eta)q_0 + 4\sqrt{1+\eta}\sqrt{s}\sqrt{\eta}q_z - \frac{8}{3}\eta s + \frac{1}{9}s - \mathbf{q}^2 - m_N^2 + i\epsilon\right]$$

$$\Psi_i(p_0, p, q_0', q') \to \Psi_i(p_0, p, q_0, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|).$$

For any $t = -Q^2 > -Q^2_{min} = 2/3\sqrt{s}(3m_N - \sqrt{s})$ the pole of G_1' on q_0 crosses the imaginary q_0 axis and appears in the third quadrant.

Beyond the SA:

2. Additional term from residue inside the contour of integration

Using the Cauchy theorem, one can transform the integrals over p_0, q_0 as follows:

$$\int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dq_0 \int_{0}^{\infty} dq \int_{-1}^{1} dy \ldots f(p_0, q_0, p, q, x, y) = \qquad (4)$$

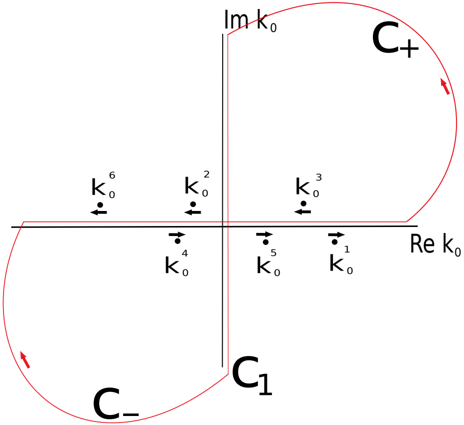
$$- \int_{-\infty}^{\infty} dp_4 \int_{-\infty}^{\infty} dq_4 \int_{0}^{\infty} dq \int_{-1}^{1} dy \ldots f(ip_4, iq_4, p, q, x, y)$$

$$+2\pi \operatorname*{Res}_{q_0 = q_0^{(2)}} \int_{-\infty}^{\infty} dp_4 \int_{q_{min}}^{q_{max}} dq \int_{y_{min}}^{1} dy \ldots f(ip_4, q_0^{(2)}, p, q, x, y),$$

where $(\ldots)$ means the two-fold integral $\int_0^\infty dp \int_{-1}^1 dx$ and

$$q_0^{(1,2)} = \frac{\sqrt{s}}{3}(1+6\eta) \pm \sqrt{4\eta(1+\eta)s - 4\sqrt{s}\sqrt{\eta}\sqrt{1+\eta}qy + \mathbf{q}^2 + m_N^2} \qquad (5)$$

are the simple poles of the propagator G_1'.


Im k_0
C_+
k_0^6
k_0^2
k_0^3
Re k_0
k_0^4
k_0^5
k_0^1
C_1
C_-

Beyond the SA:

3. Final function arguments transformation

Remembering that the BSF solutions are known for real values of q_4 only, the following assumption was made:

$$\Psi(p_0, p, q_0', q') \to g(p_0, p)\, \tau[(\frac{2}{3}\sqrt{s} + q_0^{(2)})^2 - \bar{\mathbf{q}}'^2]\, \Phi(0, \bar{q}'),$$

where value $\bar{q}'$ is obtained using (3) with $q_0 = q_0^{(2)}$.

The expansion of the function $\Phi(q_4', q')$ up to the first order of the parameter η:

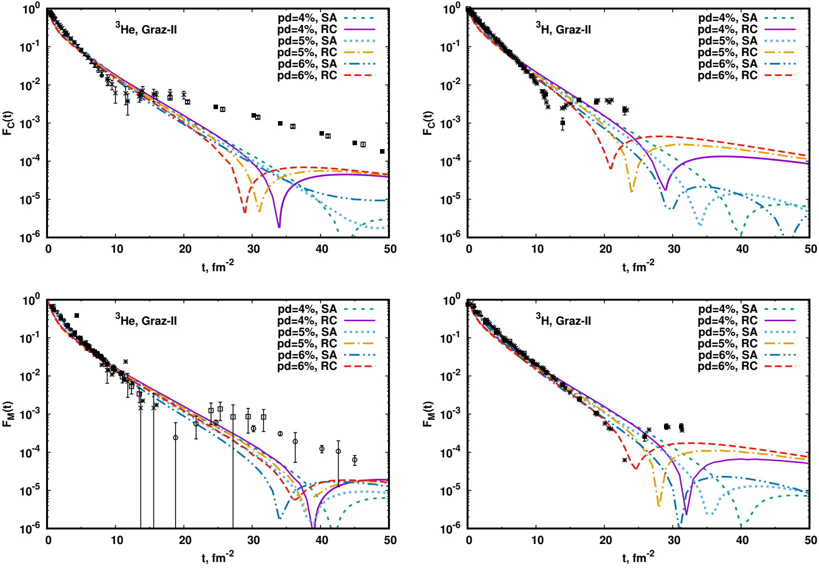
$$\Phi(iq_4', q') = \Phi(iq_4, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|) + \left[C_{q_4}\frac{\partial}{\partial q_4}\Phi_j(iq_4, q)\right]_{q=|\mathbf{q}-\frac{2}{3}\mathbf{Q}|} + \left[C_q\frac{\partial}{\partial q}\Phi_j(iq_4, q)\right]_{q=|\mathbf{q}-\frac{2}{3}\mathbf{Q}|},$$

where

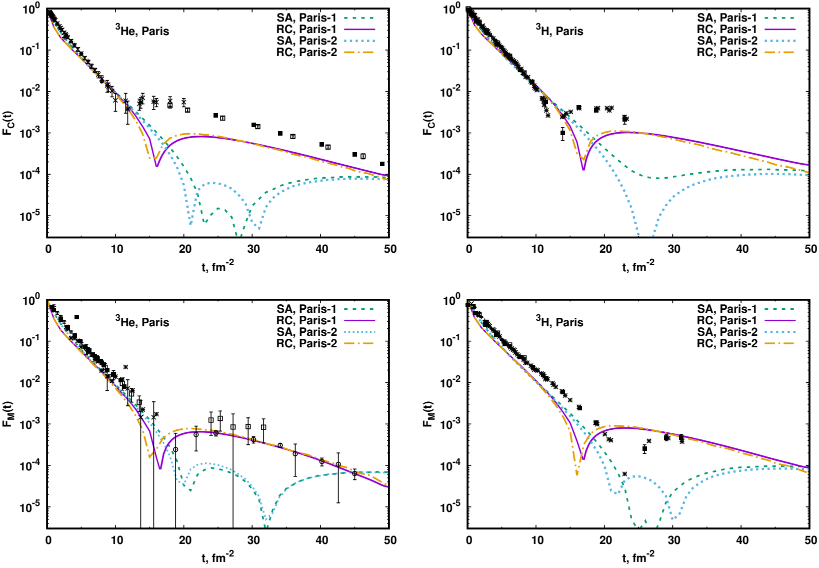
$$C_{q_4} = -i\left(2i\eta q_4 - 2\sqrt{\eta}\sqrt{1+\eta}\, q\cos\theta_{qQ} + \frac{2}{3}\sqrt{\eta}Q\right),$$

$$C_q = \left(2\eta q\cos\theta_{qQ} - 2i\sqrt{\eta}\sqrt{1+\eta}\, q_4 - \frac{2}{3}(\sqrt{1+\eta} - 1)Q\right)\cos\theta_{qQ}.$$

Graz-II relativistic kernel



Paris relativistic kernel



Summary

- the relativistic three-nucleon vertex functions were found by solving the BSF system of equations
- the charge and magnetic EM form factors of the $3N$ systems were calculated
- the static approximation and relativistic corrections were investigated

How to improve

- beyond the RIA: two- and three-nucleon EM currents
- no $3N$ forces - the phenomenological $2N$ kernel from the $2N$ observables is used (not included the $3N$ observables)

The way to investigate

- the unbound $3N$ systems: $3N$, Nd scattering states
- the $4N$ Yakubovsky equation with $2N$ BS solution